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Subsistence Consumption and Natural Resource Depletion: Can resource-rich low-income countries realize sustainable consumption paths?

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Abstract

This contribution is concerned with efficient use of a resource if households are characterized by Stone-Geary preferences with a minimum subsistence level of consumption. Subsistence consumption implies particular minimum requirements for initial endowments with reproducible man-made capital and resources. If these are not met, the economy is not able to cover subsistence consumption such as nutrition. Focusing on the steady state, we find that the equilibrium can be governed by zero or positive growth. The latter occurs if the rate of exogenous technical change exceeds the rate of time preference. In the former case, we can show that Hartwick's investment rule applies in a steady state. Finally, we calibrate the model for developing but resource-rich countries and trace the full dynamic development of the economy. Furthermore, we evaluate this full adjustment process regarding several sustainability indicators.

Keywords

Automation, Minimum subsistence level of consumption; Resource depletion; Resource-rich low-income countrie

JEL Classifications

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1 Introduction

This contribution reconsiders the continuous time Dasgupta-Heal-Solow-Stiglitz (Dasgupta and Heal 1974, Solow 1974 and Stiglitz 1974, DHSS from here on) model extended to include exogenous technical change and depreciation in reproducible man-made capital as has been proposed by Dixit et al. (1980, DHH from here on). We search for a closed-form solution to the model that maximizes a utilitarian criterion that allows for subsistence consumption in the spirit of Stone (1954) and Geary (1950). Subsistence consumption can be associated with the standard of living covering the mental and physical needs of life (Sharif 1986). Moreover, subsistence consumption also corresponds to the concept of the poverty line that can be used to quantify the proportion of society's absolutely poor members. In other words, at low-income levels, the propensity to save is low since biological needs such as nutrition have to be covered. As a consequence, the elasticity of intertemporal substitution approaches zero when consumption is close to the survival level. Unsurprisingly, the requirement to cover basic needs affects the growth process of a country as well as Stone (1954) and Geary (1950) type of preferences can generate humped-shaped growth rates (King and Rebelo (1989)).

This contribution also adds to the discussion on sustainability in the presence of scarce resources. A frequently cited concept of sustainability originating from the Brundtland Report (WCED 1987) puts forward the aim to *"make development sustainable to ensure that it meets the needs of the present without compromising the ability of future generations to meet their own needs"*. Stone-Geary preferences ensure that consumption never falls short of a minimum subsistence level. If one is willing to interpret *meeting needs* as guaranteeing a particular minimum consumption or basic needs to which the Brundtland Report also refers, our results are helping to judge whether sustainable development in the presence of necessary resources is feasible. Yet, there are other interpretations and concepts of sustainability. We contribute to the discussion by evaluating the closed-form solution regarding several sustainability indicators. We do so both theoretically and empirically in the calibration of the model to scenarios relevant to resource-rich low-income countries. We can comprehensively do this as we take account of technical change explicitly (see e.g. Pezzy 2004).

We contribute further to the existing literature in the following ways. First, we augment the DHSS model not only by technical change and depreciation which has been done elsewhere but also by the introduction of minimum subsistence consumption. It will be shown that only under certain endowment situations, i.e. requirements for initial stocks of reproducible capital and the resource, a solution to the problem exists. Third, the ability to fully characterize the dynamics and initial conditions for an economy allows us to calibrate our model to situations relevant to low-developed but resource-rich countries. This allows us to judge whether a particular endowment scenario is sufficient for a solution to the economic problem to exist and how such a solution is characterized.

Typical linearization techniques around a steady state wouldn't allow for this as the problem can not be solved for the initial values of the state variables. Second, we provide a technical contribution regarding closed-form solutions for the full dynamics of an economy using special functions such as the Gaussian hypergeometric function that allows us to fully compute the (global) complex transitional dynamics without relying on local approximation techniques around the steady-state as usually done in the literature. Finally, our approach is utilitarian by nature as we investigate households maximizing a discounted utility stream. As will be seen below, the presence of subsistence consumption leads the economy in some cases to asymptotically approach an egalitarian consumption path characterized by the generalized Hartwick (1977) investment rule formulated by Dixit et al. (1980). We, therefore, add to the literature initiated already by Solow (1974) on economic rules leading to egalitarian consumption paths motivated by utilitarian maximization problems.

Our findings are as follows. Solving the model in closed-form allows us to pin down the initial co-state variables for the dynamic optimization problem. This in turn allows for a full characterization of conditions under which a solution to the economic problem exists. Subsistence consumption implies particular minimum requirements for initial endowments with reproducible man-made capital and resources. If these are not met, the economy is not able to cover subsistence consumption. Focusing on the steady state, we find that the equilibrium can be governed by zero or positive growth. The latter occurs if the rate of exogenous technical change exceeds the rate of time preference. In the former case, we can show that Hartwick's investment rule applies in a steady state. Finally, we calibrate the model for developing but resource-rich countries and trace the full dynamic development of the economy. Furthermore, we evaluate this full adjustment process regarding several sustainability indicators.

The plan of the paper is as follows. The next section reviews literature relevant to our contribution. Section 3 lays out the economic problem that we aim to solve and Section 4 presents the solution and elaborates on the solution's existence properties. We provide a calibration of our model in Section 5 and, finally, we discuss and conclude in Section 6.

2 Review of Literature

The DHSS model has been subject to research until today by several authors. The Cobb Douglas constant returns to scale production structure with reproducible man-made capital and resource input has been employed by e.g. Antony and Klarl (2018), Benchekroun and Withagen (2011), Asheim and Buchholz (2004) and others. Mitra et al (2013) employ general constant returns to scale technology with reproducible man-made capital and resource input. Extensions of the model covering depreciation of capital and exogenous technical change have been put forward first in the DHH model. Extensions also considering endogenous resource augmenting technical change can be found in e.g. Groth (2007). Antony and Klarl (2018) introduce a minimum subsistence level of consumption in a utilitarian approach to the DHSS model without capital depreciation and technical change.

The DHSS model has been used in Hartwick's (1977) contribution and the resulting Hartwick's investment rule has been subject to ongoing research as well. It is well known, that this rule demands rents from resource extraction to be invested into a reproducible man-made stock of capital. Equivalently, the value of net investments or genuine savings equals zero at all points in time (see e.g. Hamilton and Atkinson 2006). The generalized Hartwick rule was first deduced from the DHH model and states that an egalitarian consumption path implies that the value of net investments or genuine savings is constant at all points in time but not equal to zero.

Egalitarian consumption paths and Hartwick's investment rule concerning utilitarian objectives have been put forward on the research agenda already in Solow (1974). Solow (1974) was showing that Rawls (1971) maximin criterion can be fulfilled by an egalitarian consumption path. Asheim and Buchholz (2004) analyzed undiscounted and discounted utility streams with exogenous restrictions to households' preferences as utilitarian objective functions. They work out under what conditions or restrictions maximization of the objective functions implies the economy follows Hartwick's investment rule. Antony and Klarl (2018) show that maximizing discounted utility with Stone-Geary-type preferences leads asymptotically to Hartwick's investment rule as consumption approaches its minimum subsistence level from above. We note that only a few contributions provide utilitarian criteria that justify the adoption of Hartwick's investment rule.

The asymptotic result in Antony and Klarl (2018) is to be expected as it can be shown that any competitive egalitarian path in a DHSS model with stationary technology must fulfill Hartwick's investment rule (Bucholtz et al. 2005, Withagen and Asheim 1998). Mitra (2002) already showed that Hartwick's investment rule is a necessary condition for such an egalitarian consumption path. Related to these paths, Mitra et al. (2013) provide necessary and sufficient conditions for the production technique to ensure that from any historical starting point a constant positive consumption stream results. Within a standard exhaustible resource model, Mitra (2015) proves the efficiency and uniqueness of non-trivial maximin paths. Comparable results regarding the generalized Hartwick rule in the DHH model can be found in Heijnen (2008) and Sato and Kim (2002). These findings become relevant in the present contribution for some cases where the economy approaches an asymptotically egalitarian path.

Others e.g. Asheim and Buchholz (2004) regard only paths with non-decreasing consumption as sustainable. The World Bank uses genuine savings as an indicator for sustainable development (see e.g. Hamilton and Naikal 2014).[1] To investigate the question of sustainability closer, we adopt the approach originally introduced by Weitzman (1976) and augmented by Weitzman (1997) to cover cases with exogenous technical change.

Our contribution touches upon the discussion on sustainability in the presence of non-renewable resources. It is beyond the scope of this section to fully review this strand of the literature. For a recent review regarding the relationship between sustainability, genuine savings, and Hartwick's investment rule see Hanley et al. (2015). Asheim and Buchholz (2004) regard any consumption path characterized by non-decreasing consumption as sustainable. Such paths can be derived from a discounted utilitarian maximization problem subject to restrictions exogenous to households' pref-

[1] For an overview of such approaches see e.g. Hanley et al. (2015).

erences. In contrast, Stone-Geary preferences guarantee a minimum level of consumption where the necessary restriction on consumption is already implied by preferences. As pointed out in the introduction, one might be willing to interpret consumption not falling below a minimum subsistence level to fulfill some sustainability concepts. More precisely, the question would be whether a non-monotonic consumption path that is strictly above a minimum subsistence level could be regarded as sustainable development. Holden et al. (2014) provide a broad review of the concept of sustainability and sustainable development. They give an interpretation of the Brundtland Report that regards development sustainable if basic human needs are guaranteed in an intergenerational way. Any aspiration beyond these needs can only be regarded as sustainable if long-term ecological sustainability is respected. Focusing on the extraction and consumption of non-renewable resources, one is therefore left with the question of whether the asymptotic exhaustion of such a resource combined with consumption asymptotically approaching basic needs from above is indeed satisfying this interpretation. Genuine savings is favored by the World Bank for analyzing questions related to sustainability (see e.g. Hamilton and Naikal 2014).

Pezzy (2004) criticizes some approaches to sustainability as they are not comprehensive. To be comprehensive, one needs to take into account the possibility of autonomous technical change. Our contribution contains a non-stationary technology with exogenous technical change. Our analytical solutions allow us to apply Weitzman's (1997) test on temporary sustainability and sustainable development along the whole adjustment trajectory of the economy. We do so by adopting the approach originally introduced by Weitzman (1976) and augmented by Weitzman (1997) to cover cases with exogenous technical change. Weitzman's approach has been developed within an environment with a constant interest rate. We adopt it by transferring the implied sustainability test to the case of non-constant interest rates during adjustment periods. It is not the aim of this contribution to fully answer all the questions surrounding sustainability. However, we would like to contribute to the discussion on possible answers.

Although the analyzed DHSS/DHH setting extended by subsistence consumption is complex due to a non-homothetic instantaneous utility function, it allows for a closed-form solution if one indeed exists. The present model is a natural extension of the model presented by Antony and Klarl (2018) discussing a Ramsey economy for the case where the rate of technical progress, as well as the depreciation rate on reproducible capital, is zero. From a technical point of view, the contribution is related to recent publications also using special functions to solve dynamic economic problems. Hiraguchi (2014) solves a Ramsey problem with leisure as one argument of the utility function. The solution involves the Gaussian hypergeometric function. The same function appears in Boucekkine and Ruiz-Tamarit (2008), Boucekkine et al. (2008), Ruiz-Tamarit (2008), Hiraguchi (2009) solving Lucas types models. Guerrini (2010) uses the Gaussian hypergeometric function to solve the problem of an AK Ramsey economy with logistic population growth. Regarding problems related to environmental economics, Perez-Barahona (2011) uses the Gaussian hypergeometric function to solve an AK Ramsey problem with scarce resources. The exponential Integral is found in the explicit solutions to a basic DHSS model without technical change and capital depreciation in Benchekroun and Withagen (2011). The formal representations involved in solving the problem we pose are most similar to the ones found in Boucekkine and Ruiz-Tamarit (2008).

Economically, our analysis is also related to the growth literature. Strulik (2010) and Steger (2000) solve a utility maximization problem with Stone-Geary preferences without considering natural resources necessary for production. They instead focus only on an AK type of production technology and present closed-form solutions for the entire adjustment path of the economy. Their models are nested in ours if one is setting the output elasticity of the resource equal to zero. Mathematically, we implicitly also make use of what is known as the mathematical concept of a viability kernel. As we can solve the entire dynamics of the model economy, we are also able to exactly solve the conditions regarding the initial endowment of the economy for a solution to exist. These conditions form the viability kernel (for an economic application of this concept related to resource problems see e.g. Martinet and Doyen, 2007).

3 Subsistence Consumption in the DHSS Model

In this section, we lay out the intertemporal utilitarian problem that we aim to solve. Preliminary calculations are presented that are helpful in finding a solution to the problem provided one exists.

3.1 The Problem

The Economy is populated by a mass 1 of infinitely living representative households with the following Stone-Geary intertemporal utility function

$$U_t = \int_0^\infty \frac{(C_t - \underline{C})^{1-\eta} - 1}{1-\eta} e^{-\rho t} dt, \tag{1}$$

where C_t is actually realized consumption at time t, $\underline{C}$ is the minimum subsistence level of consumption, $\eta > 0$ and $\rho > 0$ is the rate of time preference. We will refer to $C_t - \underline{C}$ as excess consumption in the sense that is taking place in excess of subsistence consumption. $\underline{C}$ can be interpreted as a given preference parameter that characterizes people's very basic needs. Satisfying basic needs is not creating any utility, only consumption exceeding this level contributes to people's well-being.

We consider a social planer to maximize households' lifetime utility given the relevant budget constraints. These constraints are given, first, by the accumulation of reproducible capital, and second, by the use of a non-renewable resource that is necessary for production. For the analysis to be conclusive, subsistence consumption needs to be exogenous also from the social planner's perspective. We will elaborate further down below on the admissible domain for $\underline{C}$ that allows for the existence of a solution to the planer's problem. This allows us to find out how the initial endowment of the economy with resources and capital relates to the admissible level of minimum subsistence consumption. It will also be shown how $\underline{C}$ is related to an (eventually asymptotic) egalitarian consumption path and how high subsistence consumption could be at maximum given initial endowments.

We assume that production is given by the aggregate Cobb-Douglas production technology

$$Y_t = K_t^{\alpha}(A_t R_t)^{1-\alpha}, \tag{2}$$

where K_t denotes the accumulated level of reproducible capital, R_t is resource use and $0 < \alpha < 1$. A_t is the level of technology which we assume to grow at rate γ, i.e. $A_t = A_0 e^{\gamma t}$, with $A_0 > 0$ as the initial level of technology. With specification (2), we have constant returns to scale with respect to capital and resource input R_t as e.g. in Benchekroun and Withagen (2011) or Asheim and Buchholz (2004). One might interpret K_t not only to represent physical but also any kind of reproducible capital as e.g. human capital. This last interpretation would allow also labor to participate in production.

Reproducible capital is produced from foregone final output with unit productivity and depreciates at a rate $\delta > 0$. The net increase in the stock of reproducible capital is therefore

$$\frac{\partial K_t}{\partial t} = \dot{K}_t = Y_t - C_t - \delta K_t. \tag{3}$$

Production requires the use of R_t units of a non-renewable resource at time t. The stock S_t of the resource develops according to

$$\dot{S}_t = -R_t. \tag{4}$$

The present value Hamiltonian for the representative household, therefore, reads as

$$H_t = \frac{(C_t - \underline{C})^{1-\eta} - 1}{1-\eta} e^{-\rho t} + \lambda_t [Y_t - C_t - \delta K_t] + \mu_t [-R_t], \tag{5}$$

where the co-states λ_t and μ_t can be interpreted as the shadow values of reproducible capital and the resource. The necessary first-order conditions for a maximum read as

$$\frac{\partial H_t}{\partial C_t} = (C_t - \underline{C})^{-\eta} e^{-\rho t} - \lambda_t = 0, \tag{6}$$

$$-\frac{\partial H_t}{\partial K_t} = \dot{\lambda}_t = -\lambda_t \frac{\partial Y_t}{\partial K_t} + \lambda_t \delta, \tag{7}$$

$$\frac{\partial H_t}{\partial R_t} = \lambda_t \frac{\partial Y_t}{\partial R_t} - \mu_t = 0, \tag{8}$$

$$-\frac{\partial H_t}{\partial S_t} = \dot{\mu}_t = 0. \tag{9}$$

(6) and (8) equate marginal utility of consumption with marginal cost of consumption in terms

of forgone capital accumulation and the marginal product of resources with the marginal cost of resource depletion. The price of the resource in terms of final output is given by $p_{R,t} = \frac{\mu_t}{\lambda_t}$. It is then easy to verify that (7) and (9) give rise to Hotelling's rule, i.e. the resource price needs to grow in optimum at a rate that equals the marginal product of capital net of depreciation.

The corresponding transversality conditions are

$$\lim_{t\to\infty} \lambda_t K_t = 0, \tag{10}$$

$$\lim_{t\to\infty} \mu_t S_t = 0, \tag{11}$$

These conditions are characterizing an optimal solution as the second derivatives of H_t with respect to controls and states are all non-positive and the Mangasarian as well as Arrow's sufficiency theorem applies. As usual, the transversality conditions imply that the limiting values of capital and resources are both zero.

The production function and (8) imply

$$Y_t = K_t \left(\frac{A_t R_t}{K_t}\right)^{1-\alpha} = K_t \left(\frac{\lambda_t(1-\alpha)}{\mu_t} A_t\right)^{\frac{1-\alpha}{\alpha}}. \tag{12}$$

This representation is useful as it provides us with intuition why an explicit solution can be found despite the complexity of the problem. (12) implies that production is of the AK type given the shadow values of capital and resources. AK models are known to provide us with explicit solutions in many cases as the behavior of the interest rate in such cases is typically analytically traceable and in many applications the marginal productivity of capital is simply constant. A constant marginal productivity can't be expected in our case. If, however, $\frac{\lambda_t}{\mu_t}$ can be expressed as a function of t alone, we can also represent the marginal productivity of capital as a function of time which is very helpful in finding the complete explicit solution to our problem.

From (9) we can easily deduce that $\mu_t = \mu_0$ for all t. Using (7) and (8), we find λ_t to develop according to[2]

$$\lambda_t = e^{\delta t}\left[\lambda_0^{\frac{\alpha-1}{\alpha}} + (1-\alpha)^{\frac{1-\alpha}{\alpha}} A_0^{\frac{1-\alpha}{\alpha}} \mu_0^{\frac{\alpha-1}{\alpha}} \frac{\alpha}{\gamma+\delta}\left(e^{\frac{1-\alpha}{\alpha}(\gamma+\delta)t} - 1\right)\right]^{\frac{\alpha}{\alpha-1}}, \tag{13}$$

which shows that the co-states can be represented as a function of t alone. Additionally, (13) is useful as it gives the development of marginal utility of consumption in excess of subsistence consumption $\underline{C}$.

[2]See Appendix A at the end of the paper for the details on the derivations.

3.2 Preliminary Calculations

The next step is now to solve for the development of the capital stock K_t. Readers not interested in the details behind the solution might skip the following and proceed directly with Proposition 1 further down below. Using (3) and (13) in (12), this stock develops as

$$\dot{K}_t = Y_t - C_t - \delta K_t = K_t\left[\left(\frac{\lambda_t(1-\alpha)}{\mu_0}A_t\right)^{\frac{1-\alpha}{\alpha}} - \delta\right] - (C_t - \underline{C}) - \underline{C}.$$

This first-order differential equation can be solved explicitly as λ_t has the explicit solution (13) which also gives excess consumption $C_t - \underline{C}$ as a function of t alone via (6). To see this, rewrite this first-order differential equation as

$$\begin{aligned} \dot{K}_t + f(t)K_t &= g(t), \\ &\text{with} \\ f(t) &= -\left[\left(\frac{\lambda_t(1-\alpha)}{\mu_0}A_t\right)^{\frac{1-\alpha}{\alpha}} - \delta\right], \\ g(t) &= -(C_t - \underline{C}) - \underline{C} = -\lambda_t^{-\frac{1}{\eta}} e^{-\frac{\rho}{\eta}t} - \underline{C}, \end{aligned} \tag{14}$$

where $-f(t)$ is the net productivity of reproducible capital at time t, i.e. $\frac{Y_t}{K_t} - \delta$. We denote the initial stock of capital at $t = 0$ by K_0. The solution to the differential equation (14) is given by

$$K_t = K_0 e^{-\int_0^t f(z)dz} + \int_0^t g(z)e^{-\int_z^t f(s)ds}dz. \tag{15}$$

Appendix B at the end of the paper shows that the compounded capital productivity net off depreciation between time z and t satisfies

$$- \int_z^t f(s)ds = -\delta(t-z) + \frac{1}{1-\alpha}\ln\left[\frac{\varphi_1 + \varphi_2\left(e^{\psi t} - 1\right)}{\varphi_1 + \varphi_2\left(e^{\psi z} - 1\right)}\right] \tag{16}$$

with

$$\varphi_1 = \left[\frac{\lambda_0(1-\alpha)A_0}{\mu_0}\right]^{\frac{\alpha-1}{\alpha}}, \quad \varphi_2 = \frac{\alpha}{\gamma+\delta}, \quad \psi = \frac{1-\alpha}{\alpha}(\gamma+\delta).$$

φ_1, φ_2 and ψ serve the purpose of simplifying the notation. From condition (8) and (12) it follows that $\varphi_1 = \frac{K_0}{Y_0}$, i.e. the inverse of the initial capital productivity. Further down below we will show that $\varphi_2 = (\lim_{t\to\infty} \frac{Y_t}{K_t})^{-1}$, i.e. the inverse of the asymptotic capital productivity. Furthermore, it will also be helpful to define $\zeta = \frac{\varphi_2 - \varphi_1}{\varphi_2} = \frac{\frac{Y_0}{K_0} - \lim_{t\to\infty}\frac{Y_t}{K_t}}{\frac{Y_0}{K_0}}$ as the relative distance of the asymptotic capital productivity from its initial value at time 0. The transitional dynamics of the model's variables are depending crucially on the value of ζ. Its deviation from zero measures the initial distance from a balanced growth path which is influencing the transitional dynamics of the economy significantly as will be shown further down below.

At this point it is also instructive to introduce the variable $x_t = e^{-\psi t}$ as an affine function of time t. As time t develops form 0 to ∞, the variable x_t ranges between 1 and 0. This makes it possible to solve the model analytically along the domain of x_t using the Gaussian hypergeometric function.

Using (16) in (15) gives the stock of capital K_t as[3]

$$
\begin{aligned}
K_t \quad &= K_0 e^{-\delta t}(1-\zeta)^{-\frac{1}{1-\alpha}}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \\
&-\frac{C_0-\underline{C}}{\psi} e^{-\delta t}(1-\zeta)^{-\frac{\alpha}{(1-\alpha)\eta}}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \int_{x_t}^{1} x_z^{-\frac{1}{\psi}\left(\frac{(\eta-1)\delta-\rho}{\eta}+\frac{\alpha-\eta}{1-\alpha}\frac{\psi}{\eta}\right)-1}(1-\zeta x_z)^{\frac{\alpha-\eta}{\eta(1-\alpha)}} dx_z \\
&-\frac{\underline{C}}{\psi} e^{-\delta t}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \int_{x_t}^{1} x_z^{-\frac{1}{\psi}\left(\delta-\frac{\psi}{1-\alpha}\right)-1}(1-\zeta x_z)^{-\frac{1}{1-\alpha}} dx_z .
\end{aligned} \tag{17}
$$

(17) traces out the entire dynamics of the economy's reproducible capital stock. The three components of (17) are representing, first, the positive contribution of the initial capital stock K_0 through production in capital accumulation, and second, the negative contributions of using production partially for consumption purposes. The latter can be divided into the part caused by consumption in excess of subsistence needs, $C_t - \underline{C}$ and a part caused by the needs to cover $\underline{C}$. Given the model's parameters, the entire path is characterized by the initial endowment K_0, ζ, and the initial consumption choice C_0. Section 4 will show how C_0 together with ζ are pined down by initial endowments. If it were by chance that $\zeta = 0$, i.e. the initial and asymptotic capital productivity coincides, we would encounter an economy that starts in steady-state right away where capital productivity, and hence the interest rate, is a constant. Unsurprisingly, the expressions in (17) would simplify a great deal if this case prevails. As the first order conditions (6) and (8) imply $\lambda_0, \mu_0 > 0$, we necessarily find $\zeta < 1$.

Appendix B at the end of the paper demonstrates that the integrals in (17) - as long as they converge - can be computed using the Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$ which has in general the integral representation

$$
{}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt. \tag{18}
$$

This integral representation is valid for $\Re(c) > \Re(b) > 0$ where $\Re(\cdot)$ denotes the real part of

[3] It is interesting to note that the integral in the second term of (17) simplifies very much in case $\alpha = \eta$. It is exactly this case that is discussed in Smith (2006) who presents a closed-form solution to the Ramsey problem for $\alpha = \eta$.

the argument and $\Gamma(\cdot)$ the Gamma function (Abramowitz and Stegun, 1964, 15.3.1). In general, ${}_2F_1(a,b;c;z)$ defined as a Gauss series (Abramowitz and Stegun, 1964, 15.1.1) converges if $\Re(c-b-a)>0$ for $|z|\leq 1$ and if $-1<\Re(c-b-a)\leq 0$ for $|z|\leq 1$ but $z\neq 1$. Comparing the integral on the right-hand side of (18) with the integrals in (17) reveals that the present case can be seen as a special case with $c-b-1=0$ or equivalent $c=b+1$.

If we apply the representation (18) to our problem, ζ will play the role of z. Admissible values for the initial co-state variables, i.e. $\lambda_0>0$ and $\mu_0>0$ (see the first order conditions (6) and (8)), imply $\zeta<1$. If λ_0 is sufficiently small and/or μ_0 is sufficiently large, it might turn out that $\zeta\leq -1$. In this case, one has to take care about how to compute the integrals in (17) or other integrals of the same type that appear further down below. This is because the integral representation (18) is an analytic continuation of the Gaussian hypergeometric function defined by a Gauss series (Abramowitz and Stegun 1972, 15.3.1). Only for the restrictions on z and $\Re(c-b-a)$ laid out above, both are identical. In general, for $z\leq -1$ and $\Re(c)>\Re(b)>0$, the integral (18) exists but the Gauss series that defines the hypergeometric function is not converging and, hence, it is not identical to the integrals that we aim to compute. In such cases, it is necessary to use analytic continuation formulas for ${}_2F_1(a,b;c;z)$ (see Abramowitz and Stegun 1972, 15.3.3 through 15.3.9). For a general discussion about this situation see Section 3.1 in Boucekkine and Ruiz-Tamarit (2008).

We, therefore, make notational use of

$$
\begin{aligned}
{}_2F_1(a,b;b+1;z) &= \frac{\Gamma(b+1)}{\Gamma(b)\Gamma(1)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt \\
&= \frac{\Gamma(b+1)}{\Gamma(b)\Gamma(1)}\int_0^1 t^{b-1}(1-zt)^{-a}dt \\
&= b\int_0^1 t^{b-1}(1-zt)^{-a}dt,
\end{aligned}
$$

where we keep in mind that $z\leq -1$ needs special attention. Here we apply the continuation of the gamma function $\Gamma(b+1)=b\Gamma(b)$ and the fact that $\Gamma(1)=1$ (Abramowitz and Stegun, 1964, 6.1.15). Inspecting (17) shows that we can apply this special case of the Gaussian hypergeometric function to both integrals. Through a suitable change in the variable of integration, the integrals ranging from x_t to 1 can be split up into two separate integrals each running from 0 to 1 and each representable by the hypergeometric function. We are ready to formulate the following proposition on the development of K_t.

Proposition 1: The optimal path for the capital stock K_t is given by

$$
\begin{aligned}
K_t \quad &= K_0 e^{-\delta t}(1-\zeta)^{-\tilde{a}_2}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\tilde{a}_2} \qquad (19)\\
&-\frac{C_0-\underline{C}}{\psi}e^{-\delta t}(1-\zeta)^{\tilde{a}_1-\tilde{a}_2}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\tilde{a}_2}\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)-x_t^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t)}{\tilde{b}_1}\\
&-\frac{\underline{C}}{\psi}e^{-\delta t}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\tilde{a}_2}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)-x_t^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)}{\tilde{b}_2},
\end{aligned}
$$

with

$$
\begin{aligned}
\tilde{a}_1 &= \frac{\eta-\alpha}{\eta(1-\alpha)}, \quad \tilde{b}_1 = 1+\frac{\alpha[(\eta-1)\gamma+\rho]}{(1-\alpha)(\delta+\gamma)\eta},\\
\tilde{a}_2 &= \frac{1}{1-\alpha}, \quad \tilde{b}_2 = 1+\frac{\alpha\gamma}{(1-\alpha)(\gamma+\delta)} > 1.
\end{aligned}
$$

Proof: Appendix B.

Note: As above, this representation allows us to trace the origins of the model's dynamics. We find a decaying influence of the initial endowment K_0 (first term). This is combined with an increasing influence of consumption in excess of its minimum subsistence level and the influence of constant subsistence consumption itself (second and third term). The special function ${}_2F_1(.)$ is involved independently of whether $\underline{C}=0$ or not.

At this point, a view words on the admissible values for the model's parameters are in order. Later on, during inspecting the transversality conditions for the present optimization problem, it will become clear that $\tilde{b}_1, \tilde{b}_2 > 1$ need to be fulfilled for the transversality conditions to hold. It is obvious that this imposes the restriction $\gamma > 0$ in case of $\tilde{b}_2$. $\tilde{b}_1 > 1$ implies further restrictions for the model's parameter. We will return to this issue further below where we deal with the transversality conditions in full detail. The variable ζ depends via φ_1 on the initial values of the co-state variables, i.e. λ_0 and μ_0. The first-order conditions (6) through (8) require $\lambda_0, \mu_0 > 0$ which in turn implies $\zeta < 1$.

With the development of the capital stock K_t at hand, we can now immediately proceed with the second input factor, the use of the resource R_t. This is particularly straightforward as $R_t = \left(\frac{\lambda_t(1-\alpha)}{\mu_0}A_t\right)^{\frac{1}{\alpha}}\frac{K_t}{A_t} = \left(\frac{p_{R,t}}{(1-\alpha)A_t}\right)^{-\frac{1}{\alpha}}\frac{K_t}{A_t}$ which follows directly from equations (12) and (8).

Proposition 2: The optimal path for resource depletion R_t is given by

$$\begin{aligned} R_t \;&=\; R_0 x_t^{-1} \\ &-\varphi_2^{-\tilde{a}_2} x_t^{-1} A_0^{-1} \left[\frac{C_0 - \underline{C}}{\psi} (1-\zeta)^{\tilde{a}_1 - \tilde{a}_2} \frac{{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta) - x_t^{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta x_t)}{\tilde{b}_1} \right. \\ &\left. + \frac{\underline{C}}{\psi} \frac{{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta) - x_t^{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta x_t)}{\tilde{b}_2} \right], \\ &\text{with} \quad R_0 = \left(\frac{p_{R,0}}{(1-\alpha) A_0} \right)^{-\frac{1}{\alpha}} \frac{K_0}{A_0}. \end{aligned} \tag{20}$$

Proof: Appendix C.

Proposition 2 gives resource depletion a weighting function of its initial value R_0 and the resources needed to cover consumption in excess of its minimum subsistence level and what is needed to cover $\underline{C}$ from $t = 0$ onward. This can be seen as x_t ranges from 1 to 0 as t passes by and the two terms involving the hypergeometric function equaling 0 for $t = 0$.

As we now know the extent of resource extraction, it is natural to proceed with the development of the economy's resource stock S_t. By assumption, $S_t = S_0 - \int_0^t R_z dz$.

Proposition 3: The resource stock S_t follows the optimal path given by

$$\begin{aligned} S_t \;&=\; S_0 - \int_0^t R_z dz, \\ &\text{with} \\ \int_0^t R_z dz \;&=\; \frac{R_0}{\psi} \frac{1-x_t}{x_t} \\ &-\varphi_2^{-\tilde{a}_2} \frac{C_0 - \underline{C}}{\psi^2} (1-\zeta)^{\tilde{a}_1 - \tilde{a}_2} \frac{1}{A_0} \left[\frac{1-x_t}{x_t} \frac{{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta)}{\tilde{b}_1} \right. \\ &\left. - \frac{{}_2F_1(\tilde{a}_1, \tilde{b}_1 - 1; \tilde{b}_1 + 1; \zeta) - x_t^{\tilde{b}_1 - 1} {}_2F_1(\tilde{a}_1, \tilde{b}_1 - 1; \tilde{b}_1 + 1; \zeta x_t)}{\tilde{b}_1 (\tilde{b}_1 - 1)} \right] \\ &-\varphi_2^{-\tilde{a}_2} \frac{\underline{C}}{\psi^2} \frac{1}{A_0} \left[\frac{1-x_t}{x_t} \frac{{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta)}{\tilde{b}_2} \right. \\ &\left. - \frac{{}_2F_1(\tilde{a}_2, \tilde{b}_2 - 1; \tilde{b}_2 + 1; \zeta) - x_t^{\tilde{b}_2 - 1} {}_2F_1(\tilde{a}_2, \tilde{b}_2 - 1; \tilde{b}_2 + 1; \zeta x_t)}{\tilde{b}_2 (\tilde{b}_2 - 1)} \right]. \end{aligned} \tag{21}$$

Proof: Appendix D.

Propositions 1 and 3 will be needed in the following section to solve for the initial co-states λ_0 and μ_0. These values are necessary to pin down initial consumption and initial resource use. The remaining results from this section are then required to trace the model's dynamics for $t > 0$.

Propositions 1 and 2 can finally be used to trace final output production Y_t given by (2) as reproducible capital and resources are the only endogenous input.

In order to complete our preliminary calculations, we look at consumption C_t.

Proposition 4: Consumption C_t follows the optimal path given by

$$C_t = (C_0 - \underline{C})(1-\zeta)^{\tilde{a}_1-\tilde{a}_2} x_t^{\frac{\gamma-\rho}{\gamma+\delta}(\tilde{a}_1-\tilde{a}_2)}(1-\zeta x_t)^{\tilde{a}_2-\tilde{a}_1} + \underline{C} \tag{22}$$

Proof: The result in Proposition 4 follows from inserting (13) into (6) and using the definition of x_t.

4 Solving the DHSS Model

The preceding section developed the dynamics for all the important quantities in the economy under consideration. In order to solve the model and to trace out the full dynamics, we need to determine the initial values for the co-state variables λ_0 and μ_0 given the initial endowment of the economy, i.e. K_0 and S_0. As will be shown, with knowledge about λ_0 and μ_0 it is possible to solve for C_0, R_0 and, hence, also Y_0. The key to this are the transversality conditions with respect to the state variable.

4.1 Transversality

The transversality conditions (10) and (11) are necessary for the solution of the problem to characterize an optimum. For the capital stock, $\lim_{t\to\infty} \lambda_t K_t = 0$ needs to be fulfilled. Furthermore, the second transversality condition demands $\lim_{t\to\infty} \mu_t S_t = \mu_0 \lim_{t\to\infty} S_t = 0$. This gives rise to the following lemma.

Lemma 1: The initial stocks of capital, K_0, and resources, S_0, need to fullfil

$$K_0 = \frac{C_0 - \underline{C}}{\psi}(1-\zeta)^{\tilde{a}_1} \frac{{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta)}{\tilde{b}_1} + \frac{\underline{C}}{\psi}(1-\zeta)^{\tilde{a}_2} \frac{{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta)}{\tilde{b}_2}, \tag{23}$$

$$S_0 = \frac{\varphi_2^{-\tilde{a}_2}}{\psi^2 A_0}\left[(C_0 - \underline{C})(1-\zeta)^{\tilde{a}_1-\tilde{a}_2} \frac{{}_2F_1(\tilde{a}_1, \tilde{b}_1-1; \tilde{b}_1+1; \zeta)}{\tilde{b}_1(\tilde{b}_1-1)} + \underline{C}\,\frac{{}_2F_1(\tilde{a}_2, \tilde{b}_2-1; \tilde{b}_2+1; \zeta)}{\tilde{b}_2(\tilde{b}_2-1)}\right] \tag{24}$$

Proof: Appendix E for (23) and F for (24).

Expression (24) implies several parameter restrictions. The first ones are about the parameter combinations $\tilde{b}_1$ and $\tilde{b}_2$. For transversality (both conditions) to hold and to be meaningful, $\tilde{b}_1 - 1 > 0$ and $\tilde{b}_2 - 1 > 0$ must be satisfied. Only then the underlying integrals behind the hypergeometric functions in (24) converge and are finite. $\tilde{b}_1 - 1 > 0$ requires $(\eta - 1)\gamma + \rho > 0$ which demands, ceteris paribus, a high rate of time preference, a high η or a low elasticity of intertemporal substitution if the rate of technical progress γ is positive. That only $\gamma \leq 0$ is in accordance with the transversality condition (24) becomes obvious as only in this case, $\tilde{b}_2 - 1 > 0$ is satisfied.

We can ask ourselves what would happen if $\tilde{b}_1 - 1 < 0$. In such a case, no finite initial capital stock K_0 would be able to meet the economy's needs to allow for a consumption path guaranteeing at least minimum subsistence consumption over the infinite time horizon. The same would apply to the case $\tilde{b}_2 - 1 < 0$. This case, however, could only prevail in case of technical regress which we ruled out right from the beginning.

Expressions (23) and (24) are two nonlinear equations in the two still unknowns λ_0 and μ_0. Both equations depend on these unknowns via the definition of φ_1 and ζ, i.e. $\varphi_1 = \left(\frac{\lambda_0(1-\alpha)A_0}{\mu_0}\right)^{\frac{\alpha-1}{\alpha}}$ and $\zeta = \frac{\varphi_2 - \varphi_1}{\varphi_2}$.

4.2 Initial Co-State Variables

The non-linearity of (23) and (24) does not allow for an explicit solution for λ_0 and μ_0. This subsection shows that the equilibrium, provided that it exists, is unique and that λ_0 and μ_0 characterize the optimum solution of our problem in terms of initial conditions for $t = 0$.

To see this, we define the following additional quantities by decomposing the transversality conditions (23) and (24)

$$K_0^+ = \frac{C_0 - \underline{C}}{\psi}(1-\zeta)^{\tilde{a}_1} \frac{{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta)}{\tilde{b}_1}, \tag{25}$$

$$\underline{K}_0 = \frac{\underline{C}}{\psi}(1-\zeta)^{\tilde{a}_2} \frac{{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta)}{\tilde{b}_2}, \tag{26}$$

$$S_0^+ = \varphi_2^{-\tilde{a}_2} \frac{C_0 - \underline{C}}{\psi^2}(1-\zeta)^{\tilde{a}_1 - \tilde{a}_2} \frac{1}{A_0} \frac{{}_2F_1(\tilde{a}_1, \tilde{b}_1 - 1; \tilde{b}_1 + 1; \zeta)}{\tilde{b}_1(\tilde{b}_1 - 1)}, \tag{27}$$

$$\underline{S}_0 = \varphi_2^{-\tilde{a}_2} \frac{\underline{C}}{\psi^2} \frac{1}{A_0} \frac{{}_2F_1(\tilde{a}_2, \tilde{b}_2 - 1; \tilde{b}_2 + 1; \zeta)}{\tilde{b}_2(\tilde{b}_2 - 1)}. \tag{28}$$

In economic terms, these four quantities have the following interpretation. K_0^+ and S_0^+ are the parts of the initial endowment with capital and resources that are necessary in optimum to allow for consumption in excess of the subsistence level, i.e. $C_t - \underline{C}$. $\underline{K}_0$ and $\underline{S}_0$ are the necessary endowments allowing for subsistence consumption $\underline{C}$ given the household's choice for the path of excess consumption $C_t - \underline{C}$. The equations just above give the optimum division of the initial endowments into the components necessary for subsistence and excess consumption. It is clear that both $K_0 - \underline{K}_0$ and $S_0 - \underline{S}_0$ need to be positive. Otherwise, initial endowments are simply insufficient to allow for subsistence consumption $\underline{C}$.

The four additional quantities are fully determined by the initial values λ_0 and μ_0 which pin down, first, their ratio via φ_1 and by that also ζ. Given ζ, the system can be solved for the levels of λ_0 and μ_0 as (55) and (27) also depend on μ_0 alone besides ζ.

From the above four definitions, it follows that

$$\frac{K_0^+}{S_0^+} = \varphi_2^{\tilde{a}_2}\psi(\tilde{b}_1 - 1)A_0(1-\zeta)^{\tilde{a}_2}\frac{{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta)}{{}_2F_1(\tilde{a}_1, \tilde{b}_1 - 1; \tilde{b}_1 + 1; \zeta)} \tag{29}$$

and

$$\frac{K_0 - \underline{K}_0}{S_0 - \underline{S}_0} = \frac{K_0 - \frac{\underline{C}}{\psi}(1-\zeta)^{\tilde{a}_2}\frac{{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta)}{\tilde{b}_2}}{S_0 - \frac{\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}\frac{1}{A_0}\frac{{}_2F_1(\tilde{a}_2, \tilde{b}_2 - 1; \tilde{b}_2+1; \zeta)}{\tilde{b}_2(\tilde{b}_2 - 1)}}. \tag{30}$$

To find ζ_0 that solves for an equilibrium, we note that this ζ_0 needs to fulfill

$$\frac{K_0 - \underline{K}_0}{S_0 - \underline{S}_0} = \frac{K_0^+}{S_0^+}. \tag{31}$$

Proposition 5: If there exists a solution ζ_0 to the equilibrium condition $\frac{K_0 - \underline{K}_0}{S_0 - \underline{S}_0} = \frac{K_0^+}{S_0^+}$ with $\zeta_0 < 1$, this solution is unique.

Proof: Appendix G making use of Lemma 1.

Appendix G proves that $\frac{K_0^+}{S_0^+}$ is decreasing and $\frac{K_0 - \underline{K}_0}{S_0 - \underline{S}_0}$ is increasing (constant) for $\underline{C} > 0$ ($\underline{C} = 0$) in ζ for $\zeta < 1$.

The properties of the right-hand side of (31) defined in (29) are $\lim_{\zeta \to -\infty} \frac{K_0^+}{S_0^+} \to \infty$ and $\lim_{\zeta \to 1} \frac{K_0^+}{S_0^+} = 0$. If subsistence consumption $\underline{C}$ would be zero, the left hand side of (31) would be constant at $\frac{K_0}{S_0} > 0$. In this case, a solution always exists. However, once $\underline{C} > 0$, there is the possibility that no solution exists. This happens whenever initial endowment with capital K_0 and the resource S_0 are too low. Sufficient initial endowments allow for $K_0 - \underline{K}_0 > 0$ and $S_0 - \underline{S}_0 > 0$ which imposes restrictions on admissible values for ζ. Define $\underline{\zeta}$ as the value for ζ that solves $K_0 - \underline{K}_0 = 0$, where $\underline{K}_0$ is given by (26). Appendix G shows that such $\underline{\zeta} < 1$ always exists for $\underline{C} > 0$. Only $\zeta \geq \underline{\zeta}$ are admissible candidates for a solution as otherwise K_0 is insufficient to guarantee subsistence consumption. On the other hand, a ζ too large may turn $S_0 - \underline{S}_0$ negative. We define $\bar{\zeta}$ as the largest possible value for

ζ that prevents $S_0 - \underline{S}_0$ from becoming non-positive (see the Appendix for the detailed derivations), i.e.

$$\bar{\zeta} =_{\zeta \leq 1} |S_0 - \underline{S}_0| =_{\zeta \leq 1} \left| S_0 - \frac{\varphi_2^{-\tilde{a}_2} \underline{C}}{\psi^2} \frac{1}{A_0} \frac{{}_2F_1(\tilde{a}_2, \tilde{b}_2 - 1; \tilde{b}_2 + 1; \zeta)}{\tilde{b}_2(\tilde{b}_2 - 1)} \right|.$$

It can be shown that $S_0 - \underline{S}_0$ is decreasing in ζ for $\underline{C} > 0$ and $\zeta \leq 1$. Hence, $\bar{\zeta}$ is bounded from above by one.

Whenever we find initial endowments and model parameters to imply $\underline{\zeta} \leq \bar{\zeta}$, we are faced with a situation that possesses a unique solution. Whenever $\underline{\zeta} > \bar{\zeta}$ prevails, there is no solution and initial endowments with K_0 and S_0 are too low. In particular, they are too low to allow for consumption to meet at least its minimum subsistence level. In case $\underline{C} \to 0$, we find $\underline{\zeta} \to -\infty$ and $\bar{\zeta} \to 1$. $\underline{\zeta}$ together with $\bar{\zeta}$ define a viability kernel as in Martinet and Doyen (2007) as they decide on whether a solution to the problem exists or not.

4.3 Transitional Dynamics

Given the initial values for the co-state variables μ_0, λ_0 and ζ_0 which are unique if a solution exists, we are able to trace out the transitional dynamics of all model's variables. We discuss below the paths for C_t, K_t, R_t and S_t. This is done by using Proposition 4 and Lemma 1 in the Propositions 1 through 3. As they can be computed explicitly, we can fully characterize the model's stable arm towards the steady state.

Applying Proposition 4 to the solution ζ_0 gives the path of consumption as

$$C_t = (C_0 - \underline{C})(1 - \zeta_0)^{\tilde{a}_1 - \tilde{a}_2} x_t^{(\tilde{a}_1 - \tilde{a}_2)\frac{\gamma - \rho}{\gamma + \delta}} (1 - \zeta_0 x_t)^{-(\tilde{a}_1 - \tilde{a}_2)} + \underline{C}. \quad (32)$$

Looking at (32) reveals that consumption can take quite different developments depending on the quantities ρ and γ as well as ζ_0. Remember that as t grows, x_t decreases from 1 towards 0. For $\zeta_0 > 0$, i.e. an initial capital productivity above its asymptotic value, we oberserve a non-monotonic consumption path with a maximum if the rate of time preference ρ exceeds the rate of technical change γ. Households are too impatient to allow for steady positive consumption growth. Instead, consumption in general first grows and peaks at $t^* = \frac{1}{\psi} \ln\left(\zeta_0 \frac{\gamma + \delta}{\rho - \gamma}\right)$ before converging asymptotically to $\underline{C}$. This can be easily affirmed by solving $\frac{\partial C_t}{\partial t} = 0$ and it is obvious that t^* only exists for the case $\rho > \gamma$.

For $\zeta_0 < 0$, we can also observe a non-monotonic behavior if $\rho < \gamma$. This case corresponds to a situation where the economy is initially endowed with a relatively large capital stock and initial capital productivity below it steady state level. Consumption then reaches a minimum at $t = t^*$ as the relatively high initial capital stock is depleted by relativly high initial consumption. From its minimum onward, consumption is permanently growing at positive rates.[4] In all other cases we observe a monotonic consumption. In particular, if $\rho < \gamma$ and $\zeta_0 > 0$, we observe a monotonic behavior of consumption. Consumption steadily increases and converges against a growth path with the positive growth rate $\frac{\gamma-\rho}{\eta}$. In case it happens that $\rho = \gamma$ applies, consumption also behaves monotonic but converges against a constant value larger than subsistence consumption.

Using Lemma 1 and Proposition 1, the development of the capital stock K_t turns out to follow (see Appendix E)

$$\begin{aligned} K_t &= \frac{C_0 - \underline{C}}{\psi}(1-\zeta_0)^{\tilde{a}_1-\tilde{a}_2} x_t^{(\tilde{a}_1-\tilde{a}_2)\frac{\gamma-\rho}{\gamma+\delta}}(1-\zeta_0 x_t)^{\tilde{a}_2} \frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta_0 x_t)}{\tilde{b}_1} \\ &+ \frac{\underline{C}}{\psi}(1-\zeta_0 x_t)^{\tilde{a}_2} \frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta_0 x_t)}{\tilde{b}_2}. \end{aligned} \tag{33}$$

The stock of reproducible capital shares qualitatively the behavior of consumption. In case of $\rho > \gamma$ it is characterized by non-monotonic behavior with a peak at in general $t \neq t^*$. Whether the peak appears earlier compared with consumption, depends on the household's preferences. It can be shown that capital peaks earlier in the (unlikely) case $\eta \leq \alpha$. In case $\eta > \alpha$, the peak can occur earlier or later.

Resource extraction can be calculated as $R_t = \left(\frac{\lambda_t(1-\alpha)}{\mu_0}A_t\right)^{\frac{1}{\alpha}} \frac{K_t}{A_t}$ by using (13) and (33). Resource extraction steadily declines as t passes by. Using (21) and (24), the resource stock S_t consequently develops according to (see Appendix F which uses Lemma 1 in Proposition 3)

$$\begin{aligned} S_t &= \frac{\varphi_2^{-\tilde{a}_2}}{\psi^2 A_0}\Bigg[(C_0-\underline{C})(1-\zeta)^{\tilde{a}_1-\tilde{a}_2} x_t^{\tilde{b}_1-1} \frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta_0 x_t)}{\tilde{b}_1(\tilde{b}_1-1)} \\ &+ \underline{C} x_t^{\tilde{b}_2-1} \frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta_0 x_t)}{\tilde{b}_2(\tilde{b}_2-1)}\Bigg]. \end{aligned}$$

Of course, S_t steadily declines and approaches 0 asymptotically.

Output is given by $Y_t = K_t^{\alpha}(A_t R_t)^{1-\alpha}$ which can now easily be computed with the results at hand. Using $R_t = \left(\frac{\lambda_t(1-\alpha)}{\mu_0}A_t\right)^{\frac{1}{\alpha}} \frac{K_t}{A_t}$, (13) and the definitions of $x_t = e^{-\psi t}$ and ζ gives

$$Y_t = \frac{K_t}{\varphi_2}(1-\zeta_0 x_t)^{-1}. \tag{34}$$

Output shares qualitatively the dynamic properties of consumption and the stock of reproducible capital. Y_t displays monotonic dynamics in case $\gamma \geq \rho$. For $\rho > \gamma$ the behavior is again non-

[4] The existence of non-monotonicity in consumption is independent of whether $\underline{C} = 0$ or $\underline{C} > 0$. The value for t^* is, however, affected by $\underline{C}$ indirectly through its dependence on ζ_0. One can show that the peak can occur earlier (later) for $\zeta_0 > 0$ ($\zeta_0 < 0$) if $\underline{C}$ is higher and the initial endowment K_0 is large. With a low K_0, the opposite happens.

monotonic with a single peak before output continuously declines. From (34) it also becomes clear that ζ_0 measures the relative distance of the steady state capital productivity from its initial position at $t=0$. Solving (34) for ζ_0 at $t=0$, gives $\zeta_0 = \frac{\frac{Y_0}{K_0}-\frac{1}{\varphi_2}}{\frac{Y_0}{K_0}}$ with $x_0=1$. Furthermore, (34) reveals that $\zeta_0 x_t$ measures the relative distance of the steady state capital productivity from its position at time t. As $x_t = e^{-\psi t}$ with $\psi = \frac{1-\alpha}{\alpha}(\gamma+\delta)$, ψ can be interpreted as a decay constant with an implied half life of $\frac{\ln(2)}{\psi} = \frac{\alpha}{1-\alpha}\frac{\ln(2)}{\gamma+\delta}$.

Given (34) and the definition of φ_2, also the net rate of return for reproducible capital can easily be computed as

$$r_t = \alpha \frac{Y_t}{K_t} - \delta = \frac{\gamma+\delta}{1-\zeta_0 x_t} - \delta. \tag{35}$$

The value of extracted resources at time t is given by $p_{R,t}R_t$ and is given by $(1-\alpha)Y_t$ as production is Cobb-Douglas. We can follow genuine savings or net investments $I_t = Y_t - C_t - \delta K_t - p_{R,t}R_t$ and the net investment rate $i_t = 1 - \frac{C_t}{Y_t} - \delta\frac{K_t}{Y_t} - \frac{p_{R,t}R_t}{Y_t}$ numerically. Its interesting limiting properties are discussed in the following subsection.

4.4 Limiting Behavior and Steady-State

Our economy approaches its steady-state as $t\to\infty$ or alternatively $x_t \to 0$. Considering the dynamic behavior of C_t, K_t and Y_t given by (32), (33) and (34) as $t\to\infty$ reveals that the growth rates of these three variables approach $\frac{\gamma-\rho}{\eta}$ in case $\gamma \geq \rho$. In this case, the three variables grow at a non-negative rate ad infinitum and the influence of subsistence consumption on the economy's growth rate vanishes asymptotically. In such a case, the economy will never be confronted with problems related to poverty. If $\gamma \leq \rho$, however, the growth rates of C_t, K_t and Y_t tend to zero as the influence of $\underline{C}$ does not vanish.[5]

The limiting behavior of C_t can easily be calculated by evaluating (32) for $x_t \to 0$. It follows that

$$\lim_{t\to\infty} C_t \begin{cases} = \underline{C}, \frac{\dot{C}_t}{C_t} \to 0 & \text{for } \rho > \gamma, \\ = (C_0 - \underline{C})(1-\zeta_0)^{\tilde{a}_1 - \tilde{a}_2} + \underline{C}, \frac{\dot{C}_t}{C_t} \to 0 & \text{for } \rho = \gamma, \\ \to \infty, \frac{\dot{C}_t}{C_t} \to \frac{\gamma-\rho}{\eta} & \text{for } \rho < \gamma. \end{cases}$$

Using (33) for the capital stock as $t\to\infty$ ($x_t \to 0$) gives

[5] In this case, e.g. the variable $C_t - \underline{C}$ and the parts of K_t and Y_t not involved covering subsistence consumption grow at rate $\frac{\gamma-\rho}{\eta}$ which is negative here.

$$\lim_{t\to\infty} K_t \begin{cases} = \frac{\underline{C}}{\psi\tilde{b}_2} \equiv \underline{K}, \frac{\dot{K}_t}{K_t} \to 0 & \text{for } \rho > \gamma, \\ = \frac{C_0-\underline{C}}{\psi\tilde{b}_1}(1-\zeta_0)^{\tilde{a}_1-\tilde{a}_2} + \frac{\underline{C}}{\psi\tilde{b}_2}, \frac{\dot{K}_t}{K_t} \to 0 & \text{for } \rho = \gamma, \\ \to \infty, \frac{\dot{K}_t}{K_t} \to \frac{\gamma-\rho}{\eta} & \text{for } \rho < \gamma, \end{cases}$$

which, using (34), implies for output Y_t

$$\lim_{t\to\infty} Y_t \begin{cases} = \frac{\underline{C}}{\psi\varphi_2\tilde{b}_2} \equiv \underline{Y}, \frac{\dot{Y}_t}{Y_t} \to 0 & \text{for } \rho > \gamma, \\ = \frac{C_0-\underline{C}}{\psi\varphi_2\tilde{b}_1}(1-\zeta_0)^{\tilde{a}_1-\tilde{a}_2} + \frac{\underline{C}}{\psi\varphi_2\tilde{b}_2}, \frac{\dot{Y}_t}{Y_t} \to 0 & \text{for } \rho = \gamma, \\ \to \infty, \frac{\dot{Y}_t}{Y_t} \to \frac{\gamma-\rho}{\eta} & \text{for } \rho < \gamma. \end{cases}$$

Naturally, the transversality conditions imply that $\lim_{t\to\infty} S_t = 0$ and consequently also $\lim_{t\to\infty} R_t = 0$.

Finally, we turn to genuine savings or net investment I_t. Inspecting the limiting behavior of Y_t, C_t and K_t above reveals that we find rather different limiting characteristics for I_t depending on γ and ρ. Using the definitions of ψ, φ_2 and $\tilde{b}_2$ gives

$$\lim_{t\to\infty} I_t = \lim_{t\to\infty}(Y_t - C_t - \delta K_t - p_{R,t}R_t) \begin{cases} = -\frac{\underline{C}}{\tilde{b}_2}, \frac{\dot{I}_t}{I_t} \to 0 & \text{for } \rho \geq \gamma, \\ \to \infty, \frac{\dot{I}_t}{I_t} \to \frac{\gamma-\rho}{\eta} & \text{for } \rho < \gamma. \end{cases}$$

In case $\gamma \leq \rho$ we find $\lim_{t\to\infty} I_t = -\frac{\underline{C}}{\tilde{b}_2}$ which is constant.[6] Hence, in the limit we find the generalized Hartwick rule fulfilled (Dixit et al. 1980). We note that genuine savings are asymptotically negative for $\underline{C} > 0$. We just note the result of constant genuine savings but do not want to interpret it here with respect to some sustainability criterion. We are digging deeper into the sustainability discussion in the next sections. In case $\gamma > \rho$ we find non-constant genuine savings which is an implication of the growing levels of consumption, output and reproducible capital. In the latter case, one can calculate the limiting rate of genuine savings as $\lim_{t\to\infty} \frac{I_t}{Y_t} = \alpha\frac{\gamma-\rho}{(\gamma+\delta)\eta} - (1-\alpha)$ which can be positive or negative depending on parameter values.

Investments into reproducible capital, $\dot{K}_t$, behave as follows. In case $\gamma \leq \rho$, we find $\lim_{t\to\infty} \dot{K}_t = 0$. As Y_t and K_t become asymptotically stationary in this case, we find resource extraction in the limit to decline at the same rate as the level of technology grows. Technical change exactly compensates for the necessary reduction in resource use. For $\gamma > \rho$, $\lim_{t\to\infty} \frac{\dot{K}_t}{Y_t} = \alpha\frac{\gamma-\rho}{(\gamma+\delta)\eta} > 0$. Now, technical change is that large that it leads asymptotically to positive growth in the reproducible capital stock.

Related to investments is the rate of return given in (35). From inspecting (35), one can see that

[6] Obviously, for $\underline{C} = 0$ we find $\lim_{t\to\infty} I_t = 0$ as well. In case $\gamma \leq \rho$ and zero subsistence consumption we find Hartwick's (1977) rule asymptotically at an egalitarian zero consumption path.

the net rate of return for reproducible capital approaches the rate of technical change in the limit, i.e. $\lim_{t\to\infty} r_t = \lim_{t\to\infty} \frac{\partial Y_t}{\partial K_t} - \delta = \gamma$.

4.5 Sustainability

It is clear from the preceding section how to trace consumption and genuine savings. Looking at consumption in Section 4.3, we found for $\gamma < (>) \rho$ a non-monotonic behavior with a maximum (minimum). If non-declining consumption is regarded as an indicator for sustainability, an economy in such a case would be categorized as behaving in an unsustainable way after the consumption peak or before its minimum.

Additionally, during our calibration in the next section, we apply Weitzman's (1976,1997) sustainability test on the economy's consumption pattern. For this, we need to adjust Weitzman's theoretical idea to our setting. In the original publications of Weitzman (1976,1997), an economy with a constant and given interest rate is analyzed. This simplifies the necessary computations considerably but limits, of course, applicability. As we can trace out the full transitional dynamics of the economy, we are able to adopt Weitzman's idea along the complete adjustment path with a non-constant interest rate.[7]

The idea behind this sustainability test is to compute a hypothetical consumption trajectory as a sustainability benchmark. Weitzman refers to this benchmark as the present value consumption annuity. To arrive at this benchmark, one computes a constant value for consumption that is in its present value equal to the present value of the welfare-maximizing consumption path. It is against which we compare the welfare-maximizing consumption choice of the representative household. As we take into account minimum subsistence consumption, we apply Weitzman's idea on the consumption in excess of its subsistence level.

Besides the notion for this benchmark as a present value equivalent annuity, Weitzman offers an alternative intuitive interpretation. As the benchmark is a present value, it can also be seen as a weighted average of the underlying welfare-maximizing consumption trajectory. The weight for each consumption value simply corresponds to its discount factor.

We are looking at our economy at time t and denote the present value (PV) of a constant excess consumption as $PV(\bar{C}_t - \underline{C})_t$, where we compute the present value over all points s in time running from t up to infinity. Note that $\bar{C}_t$ is the constant present value consumption annuity and it is carrying a time subscript as this constant consumption level depends on time t from which we start our computations. Consequently, the present value of welfare-maximizing excess consumption is denoted by $PV(C_s - \underline{C})_t$ for $s \in [t, \infty)$. Mathematically, the critical benchmark value for the sustainability test is obtained by equating $PV(\bar{C}_t - \underline{C})_t$ with $PV(C_s - \underline{C})_t$ and solving for $\bar{C}_t$.

According to Weitzman (1976,1996), sustainability at time t is given if $C_t - \underline{C} \leq \bar{C}_t - \underline{C}$ whereas

[7] Weitzman states in a footnote that other discount rates besides a constant interest rate could be considered in his approach although he "would hate to be the one who has to make such recalculations in practice" (Weitzman, 1997, fn. 6, p. 6).

we can speak of sustainable development if the whole trajectory satisfies $C_s - \underline{C} \leq \bar{C}_s - \underline{C}$ for all $s \in [t, \infty)$. Weitzman's intuition behind $\bar{C}_t$ is that it represents an annuity equivalent that is identical in its PV to C_s and is by construction never declining. $\bar{C}_t$ is then a hypothetical consumption level from time t onward, while the economy was following its welfare-maximizing consumption path up to time t.

Given the trajectory for the interest rate used for discounting, an exchange between welfare-maximizing consumption and its present value annuity would be possible. As Weitzman notes, however, such an exchange is in general hypothetical and does not need to be attainable at the economy-wide level. It is guaranteed attainable just at the margin, i.e. for an infinitesimal small consumer not able to influence developments in the interest rate. In our setting, attainability of $\bar{C}_t$ depends on the parameters of the model. We show below that attainability is not always guaranteed in our setting.

Appendix H at the end of the paper shows that we can compute $\bar{C}_t - \underline{C}$ as

$$\bar{C}_t - \underline{C} \quad = \quad (1-\zeta_0)^{\tilde{a}_1 - \tilde{a}_2}(C_0 - \underline{C})\frac{\tilde{b}_2 - 1}{\tilde{b}_1 - 1}\frac{{}_2F_1(\tilde{a}_1 - 1, \tilde{b}_1 - 1; \tilde{b}_1; \zeta_0 x_t)}{{}_2F_1(\tilde{a}_2 - 1, \tilde{b}_2 - 1; \tilde{b}_2; \zeta_0 x_t)} x_t^{(\tilde{a}_1 - \tilde{a}_2)\frac{\gamma-\rho}{\gamma+\delta}}, \qquad (36)$$

where C_0 is initial (welfare-maximizing) consumption at time 0.

For $t \to \infty$, we arrive at the steady state where we find

$$\lim_{t\to\infty} \bar{C}_t - \underline{C} \quad = \quad \begin{cases} 0 & \text{for } \rho > \gamma, \\ (1-\zeta_0)^{\tilde{a}_1 - 1}(C_0 - \underline{C})\frac{r}{r-g} & \text{for } \rho = \gamma, \\ (1-\zeta_0)^{\tilde{a}_1 - 1}(C_0 - \underline{C})\frac{r}{r-g}\lim_{t\to\infty} x_t^{(\tilde{a}_1 - \tilde{a}_2)\frac{\gamma-\rho}{\gamma+\delta}} & \text{for } \rho < \gamma. \end{cases} \qquad (37)$$

where $r = \gamma$ is the steady state interest rate and $g = \frac{1}{\eta}(\gamma - \rho)$ is the steady state growth rate of the economy.[8]

From Section 4.4 we know that optimal consumption behaves as

$$\lim_{t\to\infty} C_t - \underline{C} \quad = \quad \begin{cases} 0 & \text{for } \rho > \gamma, \\ (1-\zeta_0)^{\tilde{a}_1 - 1}(C_0 - \underline{C}) & \text{for } \rho = \gamma, \\ (1-\zeta_0)^{\tilde{a}_1 - 1}(C_0 - \underline{C})\lim_{t\to\infty} x_t^{(\tilde{a}_1 - \tilde{a}_2)\frac{\gamma-\rho}{\gamma+\delta}} & \text{for } \rho < \gamma. \end{cases} \qquad (38)$$

Comparing actual consumption (38) as $t \to \infty$ with the benchmark given by (37), shows that Weitzman's sustainability criterion is fulfilled in steady state in all the cases considered. This is unsurprising from the mathematical perspective as $\rho \leq \gamma$ is equivalent to $g \geq 0$. It is also unsurprising from the intuitive perspective as the steady state is either characterized by constant or increasing

[8] The parameter restrictions discussed in Section 3.2 ensure that $r - g > 0$.

consumption. Whether such a conclusion holds along the whole adjustment trajectory, i.e. comparing $C_t - \underline{C}$ with (36) for all t is a numerical case-specific question that we consider in the calibration during the following section.

Before proceeding with our calibration, we need to investigate the issue of the present value annuity's attainability. As mentioned already above, attainability might not be given and this could render the test at least questionable from the practical perspective. We might consider defining the sustainability benchmark differently and in an attainable way. Define $\underline{C}_t^{max}$ as the maximum constant level of consumption from t onward. This consumption level would be equal to the maximum subsistence level that is just affordable for the economy given its endowment K_t and S_t.

Alternatively, $\underline{C}_t^{max}$ can also be viewed as welfare-maximizing consumption for the case $\eta \to \infty$ if a solution to the problem exists. Intertemporal utility (1) then corresponds to the maximin criterion resulting in constant consumption maximizing welfare. If a solution exists (Lemma 1), $\underline{C}_t^{max}$ is also attainable. Appendix H shows that

$$\underline{C}_t^{max} - \underline{C} = (1-\zeta_0)^{\tilde{a}_1-\tilde{a}_2}(C_0-\underline{C})\frac{\tilde{b}_2}{\tilde{b}_1}\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta_0 x_t)}{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta_0 x_t)} x_t^{(\tilde{a}_1-\tilde{a}_2)\frac{\gamma-\rho}{\gamma+\delta}}. \tag{39}$$

The maximum constant level of consumption from $t=0$ onward can be computed as $\underline{C}_0^{max} = \psi K_0(1-\zeta_0)^{-\tilde{a}_2}\frac{\tilde{b}_2}{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta_0)}$.

Comparing $\underline{C}_t^{max}$ and $\bar{C}_t$ in (36) reveals that

$$\frac{\underline{C}_t^{max}-\underline{C}}{\bar{C}_t-\underline{C}} = \frac{\tilde{b}_1-1}{\tilde{b}_1}\frac{\tilde{b}_2}{\tilde{b}_2-1}\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta_0 x_t)}{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta_0 x_t)}\frac{{}_2F_1(\tilde{a}_2-1,\tilde{b}_2-1;\tilde{b}_2;\zeta_0 x_t)}{{}_2F_1(\tilde{a}_1-1,\tilde{b}_1-1;\tilde{b}_1;\zeta_0 x_t)}. \tag{40}$$

$\frac{\underline{C}_t^{max}-\underline{C}}{\bar{C}_t-\underline{C}}$ can be larger or smaller than unity depending on the model's parameters. A value below unity corresponds to the case where Weitzman's present value annuity is unattainable as it would exceed the maximum possible constant consumption $\underline{C}_t^{max}$. We can show analytically that this a relevant case as $t \to \infty$ as

$$\lim_{t\to\infty}\frac{\underline{C}_t^{max}-\underline{C}}{\bar{C}_t-\underline{C}} = \frac{\tilde{b}_1-1}{\tilde{b}_1}\frac{\tilde{b}_2}{\tilde{b}_2-1},$$

which is below unity for $\gamma > \rho$ (see Appendix H).

In case $\bar{C}_t$ is attainable, it is very well possible that $\underline{C}_t^{max} > \bar{C}_t$. If this happens, Weitzman's present value annuity is not the maximum possible constant consumption. A higher value for consumption would be permanently attainable given the economy's endowment.

Given this section's discussion surrounding attainability of sustainable benchmarks, we conduct a test in addition to Weitzman's original test by using $\underline{C}_t^{max}$ instead of $\bar{C}_t$. We regard a development as sustainable for all instances in time for which $C_t - \underline{C} \le \underline{C}_t^{max} - \underline{C}$ holds.

The discussion until now focused on the social planer problem. One could also ask what type policy would implement a constant consumption profile $\underline{C}_t^{max}$ for $t \in [0,\infty)$. This requires our problem to be interpreted as a problem of a representative household that internalizes its influence on the total resource stock S_t. One possible policy is a consumption tax with a time varying rate and an instantaneous redistribution of the tax revenues. The household does not internalize the consequences of its consumption choice on this transfer payment. To derive the behavior of such a tax consider the modified Hamiltonian

$$H_t = \frac{(C_t - \underline{C})^{1-\eta} - 1}{1-\eta} e^{-\rho t} + \lambda_t [Y_t - p_t C_t - \delta K_t + E_t] + \mu_t [-R_t],$$

where p_t is the price of final output used for consumption including the tax and E_t the transfer payment from the tax collecting government at time t. E_t is exogenous to the individual representative household. This modification only affects one of the first order conditions. (6) would now read as

$$\frac{\partial H_t}{\partial C_t} = (C_t - \underline{C})^{-\eta} e^{-\rho t} - \lambda_t p_t = 0.$$

If the consumption tax would be set to induce a price development $\frac{\dot{p}_t}{p_t} = r_t - \rho$, the household would voluntarily chose for constant consumption equal to $\underline{C}_t^{max}$. This tax policy would prevent e.g. a peaking consumption profile as it reduces the incentive for an initially increasing consumption path. As the tax drives up consumption prices p_t in the future, households are not longer foregoing present for the sake of higher future consumption.[9]

5 Implications for Resource-Rich Economies

This section uses the above findings to analyze the full adjustment path of the model economy calibrated to the situation of resource-rich low-income economies. Given that we can pin down the initial conditions for the solution to the problem, we can calibrate the model using recent World Bank data on endowments with produced and natural capital.

5.1 Calibration

Regarding households' preferences, ρ, η, and $\underline{C}$ need to be specified. The rate of time preference is a parameter that is frequently calibrated. We feel that an extensive discussion on this parameter's value is not necessary. We will chose $\rho = 0.03$ which seems to be a common choice also used in e.g. Benchekroun and Withagen (2011).

There exist some contributions to the literature that calibrate the type of Stone-Geary utility function that is used in the present context. Achury et al. (2012) calibrate an intertemporal utility function

[9] This result holds as only the first-order condition (6) is affected but not (7)-(9). The mentioned tax policy then only alters the resulting Keynes-Ramsey rule to produce 0 consumption growth as $\frac{\dot{\lambda}_t}{\lambda_t} = -(r_t - \rho)$ still holds. The transfer payment E_t then allows for capital accumulation that gives rise to $\underline{C}_t^{max}$ as the optimal constant consumption choice.

identical to the present one in (1) for the US and use $\eta = \frac{1}{0.23}$ which is roughly equal to 4.3. They refer to their choice of η as a standard choice in the portfolio literature. Ogaki et al. (1996) provide estimates for $\frac{1}{\eta}$ ranging from 0.569 up to 0.646 corresponding to η decreasing from about 1.68 down to 1.55. Alavarez-Pelaez and Diaz (2005) are calibrating η in a range from 1.5 up to 2.5 in their application of Stone-Geary preferences. Ravn et al. (2006, 2008) analyze the influence of subsistence points such as subsistence consumption on the dynamics of macroeconomic development in general. Despite this, their specification for intertemporal utility is in accordance with the present situation. During calibration of their models, they use a value of 2 for η. Regarding the choices for η, we follow Ravn et al. (2006, 2008) with a value of 2. This is an intermediate value that is in between what has been used in Alavarez-Pelaez and Diaz (2005) and Achury et al. (2012).

Regarding our model with a constant $\underline{C}$, we consider the absolute poverty used be the World bank.[10] As of today, the threshold for extreme absolute poverty is set at 1.90 US $ at 2011 prices at purchasing power parity (PPP) a day available to an individual for covering basic needs (Ferreira et al. 2016). By now, this is considered to apply to low-income countries. [11] We convert these numbers into yearly values at prices of 2014. We do so as we are using below the most recent numbers on resource endowments available for 2014. This gives a poverty line of 730.56 US$ at PPP.[12]

The output elasticity of resource use R_t is, given the Cobb-Douglas production technology (2), equal to the share α_R of natural resource rents in GDP. Data on the share of non-renewable resource rents in GDP is available from the World Bank.[13]

Table 1 provides a summary of the data for different groups of countries classified according to the country's level of income. It is visible that the resource dependence increases as income decreases. Resources seem to be most important for low-income countries. We, therefore, focus on this particular group in the following.

As we will see below, the labor income share in GDP will be necessary as well for our calibration. Numbers for the labor income share in GDP in 2014 were taken from the Penn World Tables 9.0. For the labor share, we cannot observe a clear pattern and observe values on average a little bit above

[10] Values for subsistence consumption have also been proposed in Koulovatianos et al. (2007) and Atkeson and Ogaki (1996) which have been used also in Achury et al. (2012) and Ogaki et al. (1996). These numbers, however, reflect very specific countries which don't seem to be in accordance with our analysis. Additionally, investigating poverty lines in this context is interesting as they influence economic policy initiatives, especially in low-income countries (see e.g. the United Nation's Sustainable Development Goal on poverty, https://www.un.org/sustainabledevelopment/).

[11] Additionally, the World Bank recently introduced two additional poverty lines applying to lower- and upper- middle-income countries at 3.20 US $ and 5.50 US $ per day at 2011 prices and PPP. For the calculation behind these numbers see Joliffe and Prydz (2016) furthermore provide an absolute poverty level for high-income countries at 21.70 US $ per day at 2011 prices and PPP.

[12] Price changes are taken into account by using the implicit GDP deflator obtained by dividing the time series for GDP at PPP valued at constant and current prices for low-income countries available at https://data.worldbank.org/indicator/NY.GDP.PCAP.PP.KD and https://data.worldbank.org/indicator/NY.GDP.PCAP.PP.CD. This results in a growth in prices of 5.34% between 2011 and 2014.

[13] Data are available from the World Bank Data Base at https://data.worldbank.org/indicator/NY.GDP.TOTL.RT.ZS. For the details on how the numbers are derived see World Bank (2011). Natural resource rents are the sum of oil, natural gas, coal (hard and soft), mineral, and forest rents.

	resource rents' share in GDP, α_R		labor income share in GDP, α_L	
	number countries	2010-2016	number countries	2014
Low-income	34	13.15	15	51.30
Lower-middle income	47	5.92	26	52.87
Upper-middle income	56	6.29	37	47.94
High-income	79	1.90	55	52.79
World	216	3.38	133	51.29

Table 1: Resource rents and labor income share in GDP in %

Note: Averages of resource rents are reported in percentages of GDP over indicated period of time. Data source https://data.worldbank.org/indicator/NY.GDP.TOTL.RT.ZS. Labor income share from the Penn World Tables 9.0 (variable labsh, https://www.rug.nl/ggdc/productivity/pwt/) for 2014. Country classification in accordance with the World Bank's classification scheme available at https://datahelpdesk.worldbank.org/knowledgebase/articles/906519.

0.5 with only moderate variation.[14] We therefore calibrate the labor income share α_L at 0.51, i.e. the world's average value found in Table 1.

In the course of calibration, reasonable numbers for the initial stocks of natural resources and reproducible capital have to be found. The World Bank (2018) provides estimates for stocks of produced, natural and human capital up to 2014 in US $. This is part of a quite comprehensive cross country database on what the World Bank terms "The Wealth of Nations". Although it is clear that such a database provides estimates only, the data are the best available and can be of use for the present purpose Tabel 2 gives a summary of the data for 2014 in per capita terms. Values in int. $ at 2014 prices were calculated by the implicit PPP exchange rate obtained from GDP data in US $ at current prices and at PPP and current prices.[15]

	2014 in US$	2014 in int. $ at PPP
produced capital	1,967	4,753
human capital	5,564	13,446
natural capital (incl. land)	6,421	15,517
natural capital (excl. land)	1,236	2,987
net foreign assets	-322	-778

Table 2: Capital/resource stock estimates 2014 per capita for low-income countries

Note: World Bank (2018, Appendix B) estimates for stocks of different types of capital per capita in 2014 US $. High-income values are averaged values (weighted by population) for OECD and non-OECD high-income countries reported in World Bank (2018, p. 233). Produced capital: machinery, equipment, structures, urban land; natural capital (incl. land): energy resources (oil, natural gas, hard coal, lignite), mineral resources (bauxite, copper, gold, iron, lead, nickel, phosphate, silver, tin, zinc), timber resources, nontimber forest resources, crop land, pasture land, protected areas. natural capital (excl. land): natural capital (incl. land) less of crop land, pasture land, protected areas. Human capital estimated from expenditures on education.

For calibration, we use data on the stocks of human and produced capital along with the stock of natural resources excluding land. The latter was chosen as these types of resources correspond with

[14] The labor shares reported in Table 1 are low compared with e.g. the traditional $\frac{2}{3}$ that is frequently used. See e.g. the discussion in Karabarbounis and Neiman (2014) on the recently decreasing development of the labor income share.

[15] Data are available from the World Bank at https://data.worldbank.org/indicator/NY.GDP.PCAP.PP.CD (US $) and https://data.worldbank.org/indicator/NY.GDP.PCAP.PP.CD (PPP). For 2014, this implies an exchange rate of 2.41656 int. $ at PPP per US $.

the resources included in the World Bank data on resource rents' share in GDP. Ideally, we would like to exclude forest-related resources as well as they are renewable. Unfortunately, resource rents are not published separately for this type of resource.

The data published in World Bank (2018) on produced capital originate largely from the Penn World Tables (PWT). Produced capital is estimated thereby employing the perpetual inventory method applying country and capital good specific rates of depreciation. These rates vary between 3 and 8% per annum. We calibrate our model using $\delta = 0.05$ as an intermediate value in accordance with the PWT.[16]

From here on, $\tilde{s}$ denotes the calibrated counterpart (measured in int. \$ at PPP) of the model's real variable s. For the initial resource stock $\tilde{S}_0$ the number on natural capital (excl. land) in Table 2 has been used. The model doesn't differentiate between physical and human capital, i.e. the only other input factor besides the resource is the stock of reproducible man-made capital with initial value $\tilde{K}_0$. As pointed out in the beginning, this stock might very well be interpreted to contain human capital as well as physical or other produced capital.

In order to calibrate the initial value for $\tilde{K}_0$, we explicitly take account of both produced and human capital in Table 2. Define

$$\tilde{K}_0 = \tilde{K}_{p,0}^{\alpha_1} \tilde{K}_{h,0}^{1-\alpha_1}, \tag{41}$$

with $0 \leq \alpha_1 \leq 1$ and denote by $\tilde{K}_{p,0}$ ($\tilde{K}_{h,0}$) produced (human) capital in the sense of Table 2. $\tilde{K}_{p,0}$ is produced capital plus net foreign assets as we are interested in the implications of the countries own resources. We set $(1-\alpha_1)\alpha = \alpha_L$ where α_L is the labor income share in GDP in the sense of Table 1. The capital income share in GDP, α_K is given by the residual $1-\alpha_L-\alpha_R$ with α_R the resource rents' share in GDP in the sense of Table 1. Hence, for calibrating α_1, we can use $\alpha_1 = 1 - \frac{\alpha_L}{\alpha}$. Proceeding this way, we find the values for the output elasticities given in Table 3.

	$1-\alpha$	α	$\alpha_1 = 1-\frac{\alpha_L}{\alpha}$	$1-\alpha_1$
low-income	0.1315	0.8685	0.4128	0.5872
lower-middle-income	0.0592	0.9408	0.4579	0.5421
upper-middle-income	0.0629	0.9371	0.4558	0.5442
high-income	0.0190	0.9810	0.4801	0.5199

Table 3: Calibration values output elasticities

By using these output elasticities we arrive at the initial values for the reproducible capital stock $\tilde{K}_0$ of 8,131 int. \$ at PPP.

Before calculating the calibrated adjustment path implied by the model's dynamics, we have to make an assumption about the calibrated scenario. We chose to calibrate the model to reflect the

[16] PWT country specific depreciation rates are available at http://febpwt.webhosting.rug.nl/Home; variable code for depreciation rates: delta.

situation prevailing during 2014. We, therefore, chose $\tilde{Y}_0$ to reflect 2014 gross national income (GNI) in int. $ at PPP. GNI instead of GDP was chosen as the model considers a closed economy and we would like to match final output predicted by the model with what the real-world economy is able to produce using its own resources.[17] Consequently, we took account of net foreign assets in the stock of produced capital and focused on the countries' own produced capital stock. GNI per capita at current prices stood at 1,027 US $ and, therefore, at 2,483 int. $ at PPP.[18]

The only parameter left to be calibrated is the rate of technical change γ. We consider the cases $\gamma \in \{0.01, 0.02, 0.03, 0.04\}$. As we assume $\rho = 0.03$, the first three scenarios create a long-run growth rate of 0 for the economy, while the last scenario implies positive long-run growth. γ at 0.01 or 0.02 will be referred to as subsistence scenarios as consumption approaches $\underline{C}$ in the long run. Table 4 summarizes the calibrated scenario and the values for the model's variables.

var.	value	param.	value	param.	value
$\tilde{K}_0$	8,131	α	0.8685	δ	0.05
$\tilde{Y}_0$	2,483	ρ	0.03	γ	$\in \left\{ \begin{array}{c} 0.01, 0.02, \\ 0.03, 0.04 \end{array} \right\}$
$\tilde{S}_0$	2,986	η	2		

Table 4: Calibration values

Note: Calibration values as explained in the main text. Values for $\gamma < \rho$ reflect the subsistence scenarios. $\gamma > \rho$ is the growth scenario with positive long-run growth. All values corresponding to nominal variables are measured in int. $ at PPP in 2014.

5.2 Calibration Results

Given the values discussed in the previous section, we are ready to solve the model and trace out its dynamics. We start with the scenario $\gamma = 0.01$ which we might term the lowest growth scenario. First, we need to find ζ_0. This is straightforward as (13) and (33) imply $Y_0 = \frac{K_0}{(1-\zeta_0)\varphi_2}$ and therefore $\zeta_0 = 1 - \frac{\gamma+\delta}{\alpha}\frac{K_0}{Y_0}$. Given ζ_0, we use (25) and (26) to solve for $\frac{\mu_0}{A_0}$ and (12) at $t = 0$ for λ_0. Finally, μ_0 is identified by searching for the value for A_0 that implies $\tilde{S}_0$ being equal to the value in Table 4.

For larger values of γ, we could repeat the above steps. This would imply a different starting level of technology A_0 for each case considered. We believe that this would make scenarios less comparable. Therefore, we chose to fix $\tilde{A}_0$ at the level implied by $\gamma = 0.01$ and instead look for the starting value S_0 that would be required to match initial production with higher rates of technical change. Of course, these values are smaller than the number in Table 4. This exercise clearly demonstrates how technical change and initial resource stocks can be substituted against each other.

[17]See Asheim and Buchholz (2004) for a discussion of Net National Income in welfare measurement and its relation to Hartwick's investment rule and the DHSS model. As we consider the DHH model taking account of capital depreciation, we consider GNI.

[18]GNI is taken from the World Bank available at https://data.worldbank.org/indicator/NY.GNP.ATLS.CD, low-income countries' population is taken from World Bank (2018, p. 233).

Each of the scenarios needs to be viable, i.e. $\underline{\zeta} < \bar{\zeta}$ has to hold. We, therefore, compute $\underline{\zeta}$ and $\bar{\zeta}$ for all cases and find that a solution exists. Given calibration values for the model's parameters and initial endowments, we can also compute $\underline{\tilde{C}}^{max}$ as the nominal counterpart of $\underline{C}^{max}$, i.e. the constant level of consumption maximizing welfare using the maximin criterion. Table 5 gives a summary of the computational calibration results.

	$\gamma = 0.01$	$\gamma = 0.02$	$\gamma = 0.03$	$\gamma = 0.04$
ζ_0	0.7715	0.7335	0.6954	0.6573
$\underline{\zeta}$	0.5589	0.4732	0.3848	0.2935
$\bar{\zeta}$	0.8260	0.7980	0.7703	0.7427
$\tilde{S}_0$	2,986	2,904	2,837	2,778
$\underline{\tilde{C}}_0^{max}$	1,750	1,771	1,790	1,809
$\tilde{A}_0$	0.005243			

Table 5: Calibration results

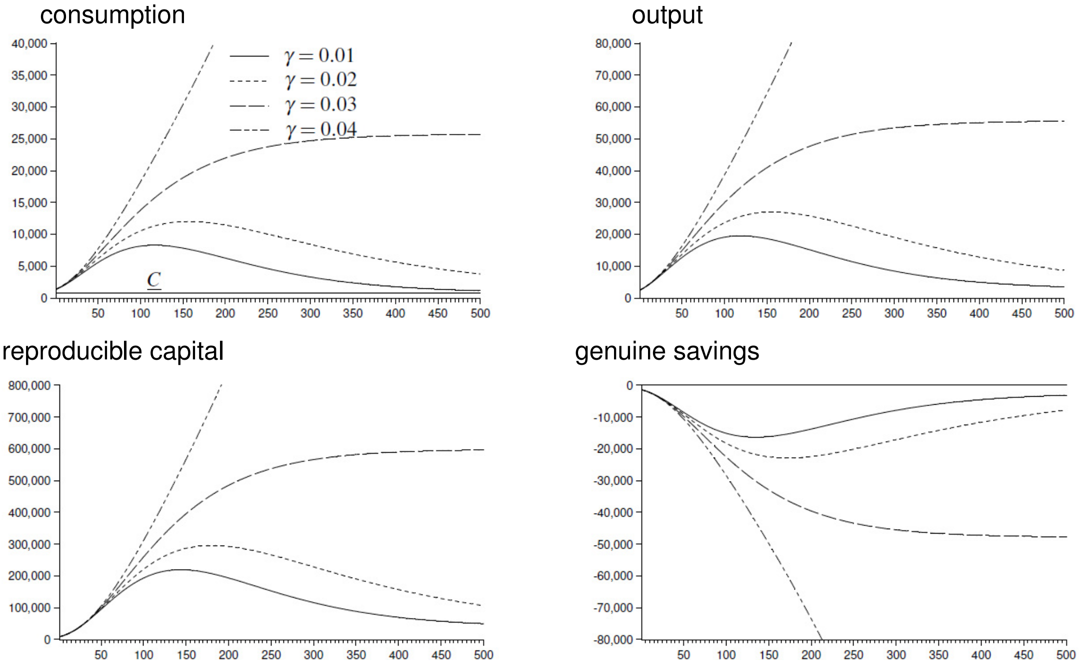
Figure 1 traces out the dynamics of consumption, output, the reproducible capital stock and genuine savings. We find that the endowment with natural and produced capital together with the implied level of technology allows consumption to go quite beyond the subsistence level during adjustment.

In the subsistence scenarios ($\gamma = 0.01$ and 0.02), annual consumption peaks at 8,271 and 11,994 2014 int. \$ at PPP respectively. Output realizes its peak at 19,509 and 26,978 2014 int. \$ at PPP. In these cases, we see consumption decline after the peak and, therefore, a sustainable development defined by non-declining consumption is not given. Non-declining consumption would, however, be possible at a constant $\underline{\tilde{C}}_0^{max}$ right from the beginning which would equal in these cases 1,750 and 1,771 int. \$ at PPP. Initial welfare-maximizing consumption is here equal to 1,351 and 1,360 int. \$ at PPP.

For the scenarios $\gamma = 0.03$ and 0.04, both, consumption and output grow monotonically. In the first case, consumption and output converge to 25,768 and 55,714 2014 int. \$ at PPP. In the latter case, both quantities grow without bounds. The stock of reproducible capital shares qualitatively the pattern of consumption and output in its development. Also the values for $\underline{\tilde{C}}_0^{max}$ in these cases are reported in Table 5 although they are less interesting here.

Finally, we see that genuine savings are negative throughout. After correcting for the part in production due to resource depletion, investments into reproducible capital do not cover depreciation. This is only affordable in the long run because technical change is substituting for these missing investments. For the cases $\gamma \leq \rho$, we find genuine savings to converge to a negative constant. For $\gamma > \rho$, genuine savings are negative as well but grow without bound.

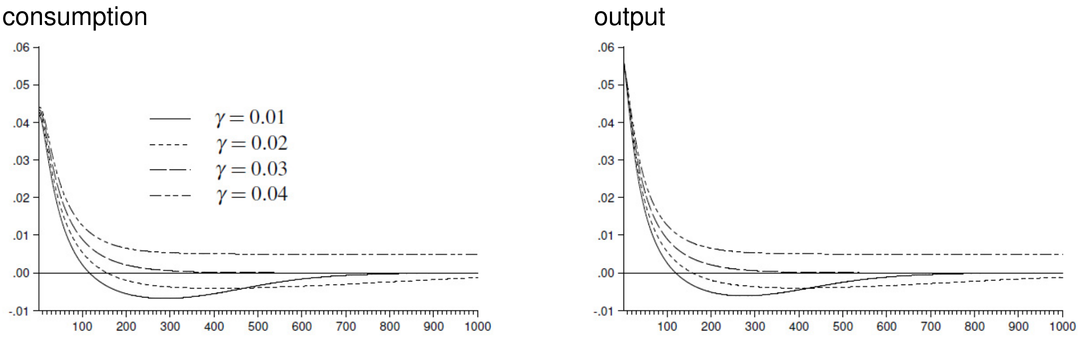
Figure 1 traces out the growth rates of consumption and output. For $\gamma < \rho$ we find the growth rate to follow non-monotonic paths. Initially, higher rates of technical change result in higher growth rates in consumption and output. However, this behavior changes later on as a higher γ translates into a higher stock of reproducible capital driving down the rate of return which is in turn decreasing



Note: Calibrated dynamics for $\tilde{C}_t$, $\tilde{Y}_t$, $\tilde{K}_t$ and genuine savings $(1-\alpha)\tilde{Y}_t - \tilde{C}_t - \delta\tilde{K}_t$ for different rates of technical change γ; all quantities in 2014 int. $ at PPP.

Figure 1: Dynamics level variables

growth.

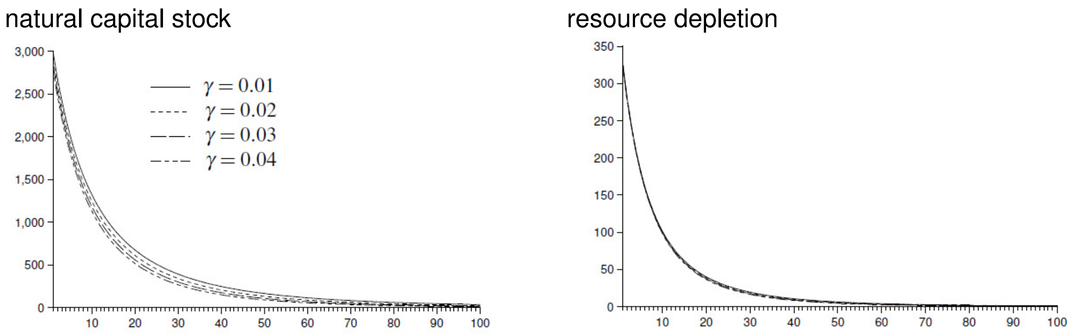


Note: Growth rates of consumption $\tilde{C}_t$ and final output $\tilde{Y}_t$ across the calibrated scenarios.

Figure 2: Growth dynamics

The stock of natural capital and resource depletion behaves monotonically as can be seen from Figure 3. Due to the chosen calibration of the four scenarios, the initial stock of natural capital is highest and identical to World Bank estimates for $\gamma = 0.01$ only. As γ increases, the initial stock $\tilde{S}_0$ declines as less resources are needed to match initial output $\tilde{Y}_0$ given the initial stock of reproducible capital $\tilde{K}_0$. Resource depletion declines monotonically as expected.

Finally, we turn to Weitzman's (1976, 1997) sustainability test by following actual consumption



Note: Stock of natural resources $\tilde{S}_t$ and resource depletion $\tilde{R}_t$ in 2014 int. $ at PPP.

Figure 3: Resource dynamics

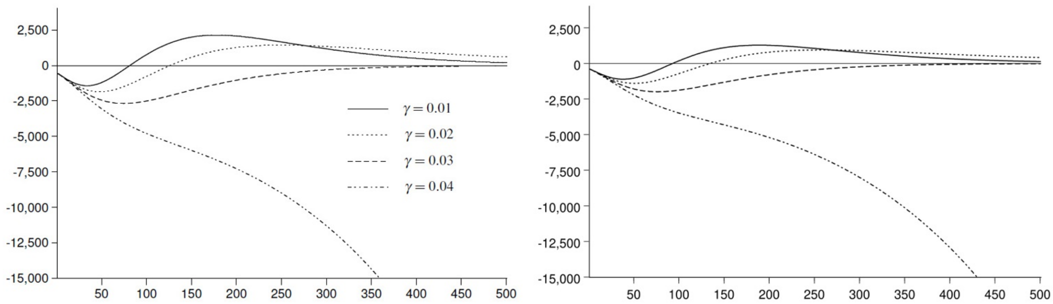
in excess of its subsistence level and the sustainability benchmark in (36) as well for the always attainable benchmark in (39). Only as long as $C_t - \underline{C} \leq \bar{C}_t - \underline{C}$ is fulfilled, i.e. actual excess consumption is below the benchmark, we find sustainability as defined by Weitzman. The alternative test uses $C_t - \underline{C} \leq \underline{C}_t^{max} - \underline{C}$.

Figure 4 plots the sustainability indicator $(C_t - \underline{C}) - (\bar{C}_t - \underline{C})$ (left panel) and $(C_t - \underline{C}) - (\underline{C}_t^{max} - \underline{C})$ (right panel). Consequently, negative values correspond to sustainability. We find the cases $\gamma \geq \rho$ to be characterized by a totally sustainable development as actual consumption always falls short of its annuity equivalent and the maximum constant consumption level. We find a sustainable development although Weitzman's present value annuity is not attainable in the limit for the case $\gamma > \rho$. In the cases $\gamma < \rho$, development is only partially sustainable according to Weitzman's criterion using both sustainability benchmarks. During adjustment, we have initially a high rate of interest as the stock of reproducible capital is low in the beginning. This is reducing initial actual consumption and lets it rise subsequently. At the same time, the high initial interest rate decreases the PV annuity equivalent and also $\underline{C}_t^{max}$. In the beginning, the first effect dominates the latter and development starts in a sustainable way. However, the order of effects is turned upside down as the economy is approaching the periods characterized by falling consumption and a lower rate of interest. This is not happening in the cases where $\gamma \geq \rho$ as γ is the limiting value for the interest rate as $t \to \infty$ and prevents r_t from falling too deep. This prevents the turnaround in the order of effects. The cases with $\gamma < \rho$ nevertheless tend to produce sustainable levels of consumption asymptotically. This reflects that consumption asymptotically becomes constant as discussed in Section 4.5.

Weitzman's test gives qualitatively the same results for both sustainability benchmarks. With the chosen parameterization, Weitzman's original specification points towards unsustainability earlier compared with the alternative using $\underline{C}_t^{max}$ as the benchmark. This corresponds to time horizons where the present value annuity actually is below the maximum possible constant consumption level.

These results stand to some extent in contrast with other sustainability indicators. Looking at the development of consumption, the difference is not as significant. In the cases $\gamma < \rho$ consumption is starting to fall a bit later compared with the point in time from which on Weitzman's test indicates unsustainability. For the cases $\gamma \geq \rho$ both indicators produce the same conclusion.

Genuine savings indicate unsustainability for the cases $\gamma > \rho$ and only asymptotically sustainability in the cases $\gamma \leq \rho$. This contrasts sharply with the outcome of Weitzman's test. This occurs as genuine savings don't take account of the exogenous technical change which is substituting for the accumulation of net wealth.



Note: Weitzman's (1997) sustainability indicator $(C_t - \underline{C}) - (\bar{C}_t - \underline{C})$ (left panel) and $(C_t - \underline{C}) - (C_t^{max} - \underline{C})$ (right panel). Negative values correspond to sustainability at the particular point in time; all quantities in 2014 int. $ at PPP.

Figure 4: Consumption and sustainability

6 Discussion and Conclusion

It is interesting to discuss why the somewhat complex set-up of this version on the DHSS/DHH model allows for a closed-form solution in terms of a special function. The key insight that can be gained from the results is that the assumption of constant returns to reproducible and natural capital leads to a particular production structure.

The reduced form for output in (34) is basically of the AK-type, i.e. $Y_t = \frac{K_t}{\varphi_2}(1 - \zeta_0 x_t)^{-1}$. Capital productivity is, of course, not constant but is given by a bounded function of time via $x_t = e^{-\psi t}$ that is traceable. This implies also a quite simple closed form for the rate of interest.

The issue of subsistence consumption is naturally tied to considerations involving Hartwick's (1977) investment rule. In the present model, this could be addressed by setting depreciation and technical change equal to zero, i.e. $\delta = \gamma = 0$. The present model incorporates this special case although the formal representation changes drastically. This can be seen by looking at the development of λ_t in (13) which is repeated for convenience

$$\lambda_t = e^{\delta t} \left[\lambda_0^{\frac{\alpha-1}{\alpha}} + (1-\alpha)^{\frac{1-\alpha}{\alpha}} \mu_0^{\frac{\alpha-1}{\alpha}} \frac{\alpha}{\gamma+\delta} \left(e^{\frac{1-\alpha}{\alpha}(\gamma+\delta)t} - 1 \right) \right]^{\frac{\alpha}{\alpha-1}}.$$

The term affected by $\delta = \gamma = 0$ is

$$\frac{\alpha}{\gamma+\delta}\left(e^{\frac{1-\alpha}{\alpha}(\gamma+\delta)t}-1\right)=(1-\alpha)\frac{e^{\frac{1-\alpha}{\alpha}(\gamma+\delta)t}-1}{\frac{1-\alpha}{\alpha}(\gamma+\delta)},$$

where $\frac{e^{\frac{1-\alpha}{\alpha}(\gamma+\delta)t}-1}{\frac{1-\alpha}{\alpha}(\gamma+\delta)t}$ is the Manly (1976) exponential transformation of t. For $\delta+\gamma\to 0$, $\frac{e^{\frac{1-\alpha}{\alpha}(\gamma+\delta)t}-1}{\frac{1-\alpha}{\alpha}(\gamma+\delta)}\to t$. This changes the formal representation of the model dramatically but allows for a closed-form solution of the dynamics. This case is treated in Antony and Klarl (2018) where it is shown that the economy asymptotically fulfills Hartwick's investment rule.

In this paper, we can provide a closed-form approach to a well-known model in resource economics. The approach uses special functions as a series of more recent contributions applies to solve dynamic problems. The advantage over the usual linearization around the steady state is that we can pin down the initial conditions for the optimal path.

The availability of the initial conditions for the co-state variables and their relation to initial endowments allows us to calibrate the model on a scenario reflecting recent estimates for endowments with produced and natural capital by the World Bank. Given these numbers, we find that low-income but resource-rich countries on average can solve the dynamic problem implied by the poverty line reflecting subsistence consumption needs.

The closed-form for the adjustment trajectory allows us to apply Weitzman's (1976,1997) approach to sustainability. The conclusion from our calibration exercise for resource-rich low-income countries is that technical change needs to be high enough to allow for an optimal and sustainable choice for consumption along with the full adjustment towards the steady state. The application of different sustainability indicators leads to different results in particular during adjustment periods.

Appendix

A: Derivation of λ_t : From (8) we know that

$$\frac{K_t}{A_t R_t} = \left(\frac{\lambda_t(1-\alpha)}{\mu_t}\right)^{-\frac{1}{\alpha}} A_t^{-\frac{1}{\alpha}}.$$

Inserting this into (7) together with $\mu_t = \mu_0$ and introducing $m_t = \lambda_t^{\frac{\alpha-1}{\alpha}}$ gives

$$\dot{m}_t = (1-\alpha)^{\frac{1}{\alpha}} \mu_0^{\frac{\alpha-1}{\alpha}} e^{\frac{1-\alpha}{\alpha}\gamma t} - \frac{1-\alpha}{\alpha}\delta m_t.$$

The solution to this differential equation can be found quite straightforward as

$$\begin{aligned} m_t &= m_0 e^{-\frac{1-\alpha}{\alpha}\delta t} + \int_0^t (1-\alpha)^{\frac{1}{\alpha}} \mu_0^{\frac{\alpha-1}{\alpha}} e^{\frac{1-\alpha}{\alpha}\gamma z} e^{-\int_z^t \frac{1-\alpha}{\alpha}\delta ds} dz \\ &= e^{-\frac{1-\alpha}{\alpha}\delta t}\left[m_0 + (1-\alpha)^{\frac{1}{\alpha}} \mu_0^{\frac{\alpha-1}{\alpha}} \int_0^t e^{\frac{1-\alpha}{\alpha}(\gamma+\delta)z} dz\right] \\ &= e^{-\frac{1-\alpha}{\alpha}\delta t}\left[m_0 + (1-\alpha)^{\frac{1}{\alpha}} \mu_0^{\frac{\alpha-1}{\alpha}} \frac{\alpha}{1-\alpha}\frac{1}{\gamma+\delta}\left(e^{\frac{1-\alpha}{\alpha}(\gamma+\delta)t} - 1\right)\right]. \end{aligned}$$

This delivers

$$\lambda_t = e^{\delta t}\left[\lambda_0^{\frac{\alpha-1}{\alpha}} + (1-\alpha)^{\frac{1-\alpha}{\alpha}} \mu_0^{\frac{\alpha-1}{\alpha}} \frac{\alpha}{\gamma+\delta}\left(e^{\frac{1-\alpha}{\alpha}(\gamma+\delta)t} - 1\right)\right]^{\frac{\alpha}{\alpha-1}}.$$

It follows that

$$\begin{aligned} \frac{\lambda_t(1-\alpha)}{\mu_0} A_t &= \left[\left(\frac{\lambda_0(1-\alpha)}{\mu_0}\right)^{\frac{\alpha-1}{\alpha}} e^{-\frac{1-\alpha}{\alpha}(\gamma+\delta)t} + \frac{\alpha}{\gamma+\delta}\left(1 - e^{-\frac{1-\alpha}{\alpha}(\gamma+\delta)t}\right)\right]^{\frac{\alpha}{\alpha-1}} \\ &= \varphi_2^{\frac{\alpha}{\alpha-1}}\left(1 - \frac{\varphi_2-\varphi_1}{\varphi_2} e^{-\psi t}\right)^{\frac{\alpha}{\alpha-1}}, \end{aligned} \tag{42}$$

with

$$\varphi_1 = \left(\frac{\lambda_0(1-\alpha)}{\mu_0}\right)^{\frac{\alpha-1}{\alpha}}, \quad \varphi_2 = \frac{\alpha}{\gamma+\delta}, \quad \psi = \frac{1-\alpha}{\alpha}(\gamma+\delta)$$

B: The capital stock K_t :

To arrive at the solution for K_t, we need first to find $-\int_z^t f(s)ds$ with $f(s) = -\left(\frac{K_s}{A_s R_s}\right)^{\alpha-1} + \delta$.

$$-\int_z^t f(s)ds = \int_z^t \left(\frac{K_s}{A_s R_s}\right)^{\alpha-1} - \delta ds = \int_z^t \left(\frac{\lambda_s(1-\alpha)}{\mu_0} A_s\right)^{\frac{1-\alpha}{\alpha}} - \delta ds.$$

Using (42) gives

$$
\begin{aligned}
-\int_z^t f(s)ds &= \int_z^t \left(\frac{\lambda_s(1-\alpha)}{\mu_0}A_s\right)^{\frac{1-\alpha}{\alpha}} - \delta ds \\
&= -\delta(t-z) + \int_z^t \left[\varphi_1 e^{-\psi s} + \varphi_2\left(1-e^{-\psi s}\right)\right]^{-1} ds \\
&= -\delta(t-z) + \frac{1}{1-\alpha}\ln\left[\frac{\varphi_1+\varphi_2\left(e^{\psi t}-1\right)}{\varphi_1+\varphi_2\left(e^{\psi z}-1\right)}\right].
\end{aligned}
$$

K_t is consequently given by

$$
\begin{aligned}
K_t &= K_0 e^{-\int_0^t f(s)ds} - \int_0^t (C_z - \underline{C}) e^{-\int_z^t f(s)ds} dz - \int_0^t \underline{C} e^{-\int_z^t f(s)ds} dz \\
&= K_0 e^{-\delta t} \left[\frac{\varphi_1 + \varphi_2(e^{\psi t} - 1)}{\varphi_1} \right]^{\frac{1}{1-\alpha}} \\
&\quad - \int_0^t \lambda_z^{-\frac{1}{\eta}} e^{-\frac{\rho}{\eta} z} e^{-\delta(t-z)} \left[\frac{\varphi_1 + \varphi_2(e^{\psi t} - 1)}{\varphi_1 + \varphi_2(e^{\psi z} - 1)} \right]^{\frac{1}{1-\alpha}} dz \\
&\quad - \int_0^t \underline{C} e^{-\delta(t-z)} \left[\frac{\varphi_1 + \varphi_2(e^{\psi t} - 1)}{\varphi_1 + \varphi_2(e^{\psi z} - 1)} \right]^{\frac{1}{1-\alpha}} dz \\
&= K_0 e^{-\delta t} \left(\frac{\varphi_2}{\varphi_1} \right)^{\frac{1}{1-\alpha}} e^{\frac{\psi}{1-\alpha} t} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi t} \right)^{\frac{1}{1-\alpha}} \\
&\quad - e^{-\delta t} \varphi_2^{\frac{1}{1-\alpha}} e^{\frac{\psi}{1-\alpha} t} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi t} \right)^{\frac{1}{1-\alpha}} \int_0^t \varphi_2^{-\frac{1}{1-\alpha}} \lambda_z^{-\frac{1}{\eta}} e^{(-\frac{\rho}{\eta} + \delta - \frac{\psi}{1-\alpha}) z} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi z} \right)^{-\frac{1}{1-\alpha}} dz \\
&\quad - \underline{C} \varphi_2^{\frac{1}{1-\alpha}} e^{-\delta t} e^{\frac{\psi}{1-\alpha} t} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi t} \right)^{\frac{1}{1-\alpha}} \int_0^t \varphi_2^{-\frac{1}{1-\alpha}} e^{\left(\delta - \frac{\psi}{1-\alpha}\right) z} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi z} \right)^{-\frac{1}{1-\alpha}} dz \\
&= K_0 e^{-\delta t} \left(\frac{\varphi_2}{\varphi_1} \right)^{\frac{1}{1-\alpha}} e^{\frac{\psi}{1-\alpha} t} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi t} \right)^{\frac{1}{1-\alpha}} \\
&\quad - e^{-\delta t} e^{\frac{\psi}{1-\alpha} t} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi t} \right)^{\frac{1}{1-\alpha}} \int_0^t \left[\frac{\mu_0}{(1-\alpha) A_0} \varphi_2^{\frac{\alpha}{\alpha-1}} e^{(\delta - \frac{\alpha}{1-\alpha} \psi) z} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2}^{-\psi z} \right)^{\frac{\alpha}{\alpha-1}} \right]^{-\frac{1}{\eta}} e^{(-\frac{\rho}{\eta} + \delta - \frac{\psi}{1-\alpha})} \\
&\quad \times \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi z} \right)^{-\frac{1}{1-\alpha}} dz \\
&\quad - \underline{C} e^{-\delta t} e^{\frac{\psi}{1-\alpha} t} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi t} \right)^{\frac{1}{1-\alpha}} \int_0^t e^{\left(\delta - \frac{\psi}{1-\alpha}\right) z} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi z} \right)^{-\frac{1}{1-\alpha}} dz \\
&= K_0 e^{-\delta t} \left(\frac{\varphi_2}{\varphi_1} \right)^{\frac{1}{1-\alpha}} e^{\frac{\psi}{1-\alpha} t} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi t} \right)^{\frac{1}{1-\alpha}} \\
&\quad - e^{-\delta t} \varphi_2^{\frac{\alpha}{(1-\alpha)\eta}} \left(\frac{\mu_0}{(1-\alpha) A_0} \right)^{-\frac{1}{\eta}} e^{\frac{\psi}{1-\alpha} t} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi t} \right)^{\frac{1}{1-\alpha}} \int_0^t e^{\left(\frac{(\eta-1)\delta - \rho}{\eta} + \frac{\alpha - \eta}{1-\alpha} \frac{\psi}{\eta} \right) z} \\
&\quad \times \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi z} \right)^{-\frac{1}{1-\alpha} + \frac{\alpha}{1-\alpha} \frac{1}{\eta}} dz \\
&\quad - \underline{C} e^{-\delta t} e^{\frac{\psi}{1-\alpha} t} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi t} \right)^{\frac{1}{1-\alpha}} \int_0^t e^{\left(\delta - \frac{\psi}{1-\alpha}\right) z} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi z} \right)^{-\frac{1}{1-\alpha}} dz
\end{aligned}
$$

Introducing $x_t = e^{-\psi t}$, $\zeta = \frac{\varphi_2 - \varphi_1}{\varphi_2}$ and noting that $\lambda_0^{-\frac{1}{\eta}} = C_0 - \underline{C}$ (condition 6) gives after simplification

$$\begin{aligned} K_t &= K_0 e^{-\delta t}(1-\zeta)^{-\frac{1}{1-\alpha}} x_t^{-\frac{1}{1-\alpha}}(1-\zeta x_t)^{\frac{1}{1-\alpha}} \\ &-(C_0-\underline{C})e^{-\delta t}(1-\zeta)^{-\frac{\alpha}{(1-\alpha)\eta}} x_t^{-\frac{1}{1-\alpha}}(1-\zeta x_t)^{\frac{1}{1-\alpha}} \int_0^t x_z^{-\frac{1}{\psi}\left(\frac{(\eta-1)\delta-\rho}{\eta}+\frac{\alpha-\eta}{1-\alpha}\frac{\psi}{\eta}\right)}(1-\zeta x_z)^{\frac{\alpha-\eta}{\eta(1-\alpha)}} dz \\ &-\underline{C} e^{-\delta t} x_t^{-\frac{1}{1-\alpha}}(1-\zeta x_t)^{\frac{1}{1-\alpha}} \int_0^t x_z^{-\frac{1}{\psi}\left(\delta-\frac{\psi}{1-\alpha}\right)}(1-\zeta x_z)^{-\frac{1}{1-\alpha}} dz \end{aligned} \tag{43}$$

Using the variable x_z, we note that $-dx_z = \psi e^{-\psi z}dz = \psi x_z dz$ and consequently, $dz = -\frac{1}{\psi}x_z^{-1}dx_z$. The region of integration changes from $[0,t]$ to $[x_t, 1]$ with $x_t \leq 1$ if we integrate over $-x_z$ instead of z. Obviously, $x_z = 1$ for $z = 0$ and $\lim_{z\to\infty} x_z = 0$.

$$\begin{aligned} K_t &= K_0 e^{-\delta t}(1-\zeta)^{-\frac{1}{1-\alpha}} x_t^{-\frac{1}{1-\alpha}}(1-\zeta x_t)^{\frac{1}{1-\alpha}} \\ &-\frac{C_0-\underline{C}}{\psi} e^{-\delta t}(1-\zeta)^{-\frac{\alpha}{(1-\alpha)\eta}} x_t^{-\frac{1}{1-\alpha}}(1-\zeta x_t)^{\frac{1}{1-\alpha}} \int_{x_t}^1 x_z^{-\frac{1}{\psi}\left(\frac{(\eta-1)\delta-\rho}{\eta}+\frac{\alpha-\eta}{1-\alpha}\frac{\psi}{\eta}\right)-1}(1-\zeta x_z)^{\frac{\alpha-\eta}{\eta(1-\alpha)}} dx_z \\ &-\frac{\underline{C}}{\psi} e^{-\delta t} x_t^{-\frac{1}{1-\alpha}}(1-\zeta x_t)^{\frac{1}{1-\alpha}} \int_{x_t}^1 x_z^{-\frac{1}{\psi}\left(\delta-\frac{\psi}{1-\alpha}\right)-1}(1-\zeta x_z)^{-\frac{1}{1-\alpha}} dx_z. \end{aligned} \tag{44}$$

Next, we turn to the integrals in (44)

$$\begin{aligned} \int_{x_t}^1 \quad & x_z^{-\frac{1}{\psi}\left(\frac{(\eta-1)\delta-\rho}{\eta}+\frac{\alpha-\eta}{1-\alpha}\frac{\psi}{\eta}\right)-1}(1-\zeta x_z)^{\frac{\alpha-\eta}{\eta(1-\alpha)}} dx_z = \\ &= \int_0^1 x_z^{-\frac{1}{\psi}\left(\frac{(\eta-1)\delta-\rho}{\eta}+\frac{\alpha-\eta}{1-\alpha}\frac{\psi}{\eta}\right)-1}(1-\zeta x_z)^{\frac{\alpha-\eta}{\eta(1-\alpha)}} dx_z - \int_0^{x_t} x_z^{-\frac{1}{\psi}\left(\frac{(\eta-1)\delta-\rho}{\eta}+\frac{\alpha-\eta}{1-\alpha}\frac{\psi}{\eta}\right)-1}(1-\zeta x_z)^{\frac{\alpha-\eta}{\eta(1-\alpha)}} dx_z \\ &= \int_0^1 x_z^{\tilde{b}_1-1}(1-\zeta x_z)^{-\tilde{a}_1} dx_z - \int_0^{x_t} x_z^{\tilde{b}_1-1}(1-\zeta x_z)^{-\tilde{a}_1} dx_z \end{aligned} \tag{45}$$

with

$$\begin{aligned} \tilde{a}_1 &= \frac{\eta-\alpha}{\eta(1-\alpha)} \\ \tilde{b}_1 &= -\frac{1}{\psi}\left(\frac{(\eta-1)\delta-\rho}{\eta}+\frac{\alpha-\eta}{1-\alpha}\frac{\psi}{\eta}\right) = 1+\frac{\alpha[(\eta-1)\gamma+\rho]}{(1-\alpha)(\delta+\gamma)\eta}. \end{aligned}$$

The integral representation of the Gaussian hypergeometric function is given by

$$_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt,$$

for $\Re(c) > \Re(b) > 0$ where $\Re(\cdot)$ denotes the real part of the argument and $\Gamma(\cdot)$ the Gamma function (see Abramowitz and Stegun, 1972, 15.3.1) where the discussion in the main text applies. The first integral in (45) therefore equals in this case

$$\begin{aligned}\int_0^1 x_z^{\tilde{b}_1-1}(1-\zeta x_z)^{-\tilde{a}_1}dx_z &= \frac{\Gamma(\tilde{b}_1)\Gamma(1)}{\Gamma(\tilde{b}_1+1)}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)\\ &= \frac{\Gamma(\tilde{b}_1)\Gamma(1)}{\tilde{b}_1\Gamma(\tilde{b}_1)}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)\\ &= \frac{1}{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta).\end{aligned}$$

The second integral in (45) can also be expressed in terms of the Gaussian hypergeometric function. We define $\tilde{x}_z = \frac{x_z}{x_t}$ which implies $dx_z = x_t d\tilde{x}_z$ and rewrite the integral as

$$\begin{aligned}\int_0^{x_t}(x_t\tilde{x}_z)^{\tilde{b}_1-1}(1-\zeta x_t\tilde{x}_z)^{-\tilde{a}_1}dx_z &= \int_0^1 (x_t\tilde{x}_z)^{\tilde{b}_1-1}(1-\zeta x_t\tilde{x}_z)^{-\tilde{a}_1}x_t d\tilde{x}_z\\ &= x_t^{\tilde{b}_1}\int_0^1 \tilde{x}_z^{\tilde{b}_1-1}(1-\zeta x_t\tilde{x}_z)^{-\tilde{a}_1}d\tilde{x}_z\\ &= x_t^{\tilde{b}_1}\frac{1}{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t).\end{aligned}$$

The second integral in (44) can also be formulated using the Gaussian hypergeometric function

$$\begin{aligned}&\int_{x_t}^1 x_z^{-\frac{1}{\psi}\left(\delta-\frac{\psi}{1-\alpha}\right)-1}(1-\zeta x_z)^{-\frac{1}{1-\alpha}}dx_z = \qquad (46)\\ &= \int_0^1 x_z^{-\frac{1}{\psi}\left(\delta-\frac{\psi}{1-\alpha}\right)-1}(1-\zeta x_z)^{-\frac{1}{1-\alpha}}dx_z - \int_0^{x_t} x_z^{-\frac{1}{\psi}\left(\delta-\frac{\psi}{1-\alpha}\right)-1}(1-\zeta x_z)^{-\frac{1}{1-\alpha}}dx_z\\ &= \frac{1}{\tilde{b}_2}\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta) - x_t^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)\right],\end{aligned}$$

with

$$\tilde{a}_2 = \frac{1}{1-\alpha},$$

$$\tilde{b}_2 = -\frac{1}{\psi}\left(\delta-\frac{\psi}{1-\alpha}\right) = \frac{(1-\alpha)\delta+\gamma}{(1-\alpha)(\gamma+\delta)} > 0,$$

where it is again required that $\tilde{b}_2 > 0$ which is here fulfilled in any case.

Using these results, the stock of capital K_t develops according to

$$\begin{aligned}K_t &= K_0 e^{-\delta t}(1-\zeta)^{-\tilde{a}_2}x_t^{-\tilde{a}_2}(1-\zeta x_t)^{\tilde{a}_2} \qquad (47)\\ &-\frac{C_0-\underline{C}}{\psi}e^{-\delta t}(1-\zeta)^{\tilde{a}_1-\tilde{a}_2}x_t^{-\tilde{a}_2}(1-\zeta x_t)^{\tilde{a}_2}\frac{1}{\tilde{b}_1}\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta) - x_t^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t)\right]\\ &-\frac{\underline{C}}{\psi}e^{-\delta t}x_t^{-\tilde{a}_2}(1-\zeta x_t)^{\tilde{a}_2}\frac{1}{\tilde{b}_2}\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta) - x_t^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)\right]\end{aligned}$$

C: Resource use R_t : Using (8), (42), (43) and (47) gives

$$
\begin{aligned}
R_t &= A_t^{-1}\left(\frac{\lambda_t(1-\alpha)}{\mu_0}A_t\right)^{\frac{1}{\alpha}} K_t \\
&= \varphi_2^{-\tilde{a}_2}(1-\zeta x_t)^{-\tilde{a}_2}A_0^{-1}e^{-\gamma t}K_t \\
&= \frac{K_0}{A_0}\varphi_2^{-\frac{1}{1-\alpha}}(1-\zeta)^{-\tilde{a}_2}x_t^{-1} \\
&\quad -\frac{C_0-\underline{C}}{\psi}\varphi_2^{-\tilde{a}_2}(1-\zeta)^{\tilde{a}_1-\tilde{a}_2}x_t^{-1}\frac{1}{\tilde{b}_1}\frac{1}{A_0}\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)-x_t^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t)\right] \\
&\quad -\frac{\underline{C}}{\psi}\varphi_2^{-\tilde{a}_2}x_t^{-1}\frac{1}{A_0}\frac{1}{\tilde{b}_2}\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)-x_t^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)\right]
\end{aligned}
$$

D: Resource stock S_t : The solution to (4) is given by

$$
S_t = S_0 - \int_0^t R_z dz + \int_0^t v dz,
$$

with

$$
\begin{aligned}
\int_0^t R_z dz &= \int_0^t \frac{K_0}{A_0}\varphi_2^{-\tilde{a}_2}(1-\zeta)^{-\tilde{a}_2}x_z^{-1}dz \\
&\quad -\frac{C_0-\underline{C}}{\psi}\varphi_2^{-\tilde{a}_2}(1-\zeta)^{\tilde{a}_1-\tilde{a}_2}\frac{1}{\tilde{b}_1}\frac{1}{A_0}\int_0^t x_z^{-1}\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)-x_z^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_z)\right]dz \\
&\quad -\frac{\underline{C}}{\psi}\varphi_2^{-\tilde{a}_2}\frac{1}{A_0}\frac{1}{\tilde{b}_2}\int_0^t x_z^{-1}\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)-x_z^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_z)\right]dz
\end{aligned}
$$

Using $-d\zeta x_z = \zeta\psi x_z dz$ and obeying that we have to integrate from ζx_t to ζ gives

$$
\begin{aligned}
\int_0^t R_z dz =& \frac{K_0}{A_0}\varphi_2^{-\tilde{a}_2}(1-\zeta)^{-\tilde{a}_2}\frac{\zeta}{\psi}\int_{\zeta x_t}^{\zeta}(\zeta x_z)^{-2} d\zeta x_z \\
&-\frac{C_0-\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}(1-\zeta)^{\tilde{a}_1-\tilde{a}_2}\frac{\zeta}{\tilde{b}_1}\frac{1}{A_0}\int_{\zeta x_t}^{\zeta}(\zeta x_z)^{-2}\Big[{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)-x_z^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_z)\Big] d\zeta x_z \\
&-\frac{\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}\frac{\zeta}{\tilde{b}_2}\frac{1}{A_0}\int_{\zeta x_t}^{\zeta}(\zeta x_z)^{-2}\Big[{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)-x_z^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_z)\Big] d\zeta x_z \\
=& -\frac{K_0}{A_0}\varphi_2^{-\tilde{a}_2}(1-\zeta)^{-\tilde{a}_2}\frac{1}{\psi}\left[1-x_t^{-1}\right] \\
&+\frac{C_0-\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}(1-\zeta)^{\tilde{a}_1-\tilde{a}_2}\frac{1}{A_0}\frac{1}{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)\left[1-x_t^{-1}\right] \\
&+\frac{C_0-\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}(1-\zeta)^{\tilde{a}_1-\tilde{a}_2}\frac{1}{A_0}\frac{\zeta^{1-\tilde{b}_1}}{\tilde{b}_1}\int_{\zeta x_t}^{\zeta}(\zeta x_z)^{\tilde{b}_1-2}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_z) d\zeta x_z \\
&+\frac{\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}\frac{1}{A_0}\frac{1}{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)\left[1-x_t^{-1}\right] \\
&+\frac{\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}\frac{1}{A_0}\frac{\zeta^{1-\tilde{b}_2}}{\tilde{b}_2}\int_{\zeta x_t}^{\zeta}(\zeta x_z)^{\tilde{b}_2-2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_z) d\zeta x_z.
\end{aligned}
$$

Using

$$
\int z^{b-2}{}_2F_1(a,b;c;z)dz = \frac{z^{b-1}}{b-1}{}_2F_1(a,b-1;c;z) + \text{constant}
$$

gives

$$
\begin{aligned}
\int_0^t R_z dz =& -\frac{K_0}{A_0}\varphi_2^{-\tilde{a}_2}(1-\zeta)^{-\tilde{a}_2}\frac{1}{\psi}\left[1-x_t^{-1}\right] \\
&+\frac{C_0-\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}(1-\zeta)^{\tilde{a}_1-\tilde{a}_2}\frac{1}{A_0}\frac{1}{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)\left[1-x_t^{-1}\right] \\
&+\frac{C_0-\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}(1-\zeta)^{\tilde{a}_1-\tilde{a}_2}\frac{1}{A_0}\frac{\zeta^{1-\tilde{b}_1}}{\tilde{b}_1(\tilde{b}_1-1)}\Big[\zeta^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)-(\zeta x_t)^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta \\
&+\frac{\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}\frac{1}{A_0}\frac{1}{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)\left[1-x_t^{-1}\right] \\
&+\frac{\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}\frac{1}{A_0}\frac{\zeta^{1-\tilde{b}_2}}{\tilde{b}_2(\tilde{b}_2-1)}\Big[\zeta^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)-(\zeta x_t)^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\Big].
\end{aligned}
$$

This gives S_t as

$$
S_t = S_0 - \int_0^t R_z dz + \nu t,
$$

where $\int_0^t R_z dz$ is given by (48) and the above parameter restrictions need to be fulfilled.

E: Transversality condition for K_t:

$$\lim_{t\to\infty} \lambda_t K_t = 0 \tag{49}$$

With (42) we find

$$\begin{aligned}
\lambda_t K_t &= e^{-\gamma t}\frac{\mu_0}{(1-\alpha)A_0}\varphi_2^{\frac{\alpha}{\alpha-1}}\left(1-\frac{\varphi_2-\varphi_1}{\varphi_2}e^{-\psi t}\right)^{\frac{\alpha}{\alpha-1}}K_t = e^{-\gamma t}\frac{\mu_0}{(1-\alpha)A_0}\varphi_2^{\frac{\alpha}{\alpha-1}}(1-\zeta x_t)^{-\frac{\alpha}{1-\alpha}}K_t \\
&= K_0 e^{-(\gamma+\delta)t}\varphi_2\varphi_1^{-\frac{1}{1-\alpha}}\frac{\mu_0}{(1-\alpha)A_0}x_t^{-\frac{1}{1-\alpha}}(1-\zeta x_t) \\
&\quad -e^{-(\gamma+\delta)t}\varphi_2^{\frac{\alpha(1-\eta)}{(1-\alpha)\eta}}\left(\frac{\mu_0}{(1-\alpha)A_0}\right)^{1-\frac{1}{\eta}}\frac{1}{\psi}x_t^{-\frac{1}{1-\alpha}}(1-\zeta x_t)\frac{1}{\tilde{b}_1}\Big[{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)-x_t^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x \\
&\quad -\underline{C}e^{-(\gamma+\delta)t}\varphi_2^{\frac{\alpha}{\alpha-1}}\frac{\mu_0}{(1-\alpha)A_0}\frac{1}{\psi}x_t^{-\frac{1}{1-\alpha}}(1-\zeta x_t)\frac{1}{\tilde{b}_2}\Big[{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)-x_t^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)\Big] \\
&= K_0\varphi_2\varphi_1^{-\frac{1}{1-\alpha}}\frac{\mu_0}{(1-\alpha)A_0}x_t^{-1}(1-\zeta x_t) \\
&\quad -\varphi_2^{\frac{\alpha(1-\eta)}{(1-\alpha)\eta}}\left(\frac{\mu_0}{(1-\alpha)A_0}\right)^{1-\frac{1}{\eta}}\frac{1}{\psi}x_t^{-1}(1-\zeta x_t)\frac{1}{\tilde{b}_1}\Big[{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)-x_t^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t)\Big] \\
&\quad -\underline{C}\varphi_2^{\frac{\alpha}{\alpha-1}}\frac{\mu_0}{(1-\alpha)A_0}\frac{1}{\psi}x_t^{-1}(1-\zeta x_t)\frac{1}{\tilde{b}_2}\Big[{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)-x_t^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)\Big]
\end{aligned}$$

As $t\to\infty$ we see $x_t\to 0$. and $x_t^{-1}\to\infty$. Rewriting $\lambda_t K_t$ as $\frac{x_t\lambda_t K_t}{x_t}$ and applying L'Hospital's rule as $x_t\to 0$ requires $\lim_{x_t\to 0}\frac{\partial x_t\lambda_t K_t}{\partial x_t}=0$. $\frac{\partial x_t\lambda_t K_t}{\partial x_t}$ computed using (50) is given by

$$\begin{aligned}
\frac{\partial x_t\lambda_t K_t}{\partial x_t} &= -K_0\varphi_2\varphi_1^{-\frac{1}{1-\alpha}}\frac{\mu_0}{(1-\alpha)A_0}\zeta \\
&\quad +\varphi_2^{\frac{\alpha(1-\eta)}{(1-\alpha)\eta}}\left[\frac{\mu_0}{(1-\alpha)A_0}\right]^{1-\frac{1}{\eta}}\frac{1}{\psi}\frac{1}{\tilde{b}_1}\zeta\Big[{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)-x_t^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t)\Big] \\
&\quad -\varphi_2^{\frac{\alpha(1-\eta)}{(1-\alpha)\eta}}\left[\frac{\mu_0}{(1-\alpha)A_0}\right]^{1-\frac{1}{\eta}}\frac{1}{\psi}(1-\zeta x_t)\frac{1}{\tilde{b}_1}\Big[-\tilde{b}_1 x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t)\Big] \\
&\quad -\varphi_2^{\frac{\alpha(1-\eta)}{(1-\alpha)\eta}}\left[\frac{\mu_0}{(1-\alpha)A_0}\right]^{1-\frac{1}{\eta}}\frac{1}{\psi}(1-\zeta x_t)\frac{1}{\tilde{b}_1}\left[-x_t^{\tilde{b}_1}\frac{\partial\,{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t)}{\partial x_t}\right] \\
&\quad +\underline{C}\varphi_2^{\frac{\alpha}{\alpha-1}}\frac{\mu_0}{(1-\alpha)A_0}\frac{1}{\psi}\frac{1}{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)\zeta \\
&\quad -\underline{C}\varphi_2^{\frac{\alpha}{\alpha-1}}\frac{\mu_0}{(1-\alpha)A_0}\frac{1}{\psi}\frac{1}{\tilde{b}_2}(1-\zeta x_t)\Big[-\tilde{b}_2 x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)\Big] \\
&\quad -\underline{C}\varphi_2^{\frac{\alpha}{\alpha-1}}\frac{\mu_0}{(1-\alpha)A_0}\frac{1}{\psi}\frac{1}{\tilde{b}_2}(1-\zeta x_t)\left[-x_t^{\tilde{b}_2}\frac{\partial\,{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)}{\partial x_t}\right]
\end{aligned}$$

Evaluating $\frac{\partial x_t\lambda_t K_t}{\partial x_t}$ at $x_t=0$ gives as long as $\tilde{b}_1-1>0$ and $\tilde{b}_2-1>0$

$$
\begin{aligned}
\left.\frac{\partial x_t \lambda_t K_t}{\partial x_t}\right|_{x_t=0} &= -K_0 \varphi_2 \varphi_1^{-\frac{1}{1-\alpha}} \frac{\mu_0}{(1-\alpha)A_0}\zeta \\
&+\varphi_2^{\frac{\alpha(1-\eta)}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0}\right]^{1-\frac{1}{\eta}} \frac{1}{\psi}\frac{1}{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta)\zeta \\
&+\underline{C}\varphi_2^{\frac{\alpha}{\alpha-1}} \frac{\mu_0}{(1-\alpha)A_0}\frac{1}{\psi}\frac{1}{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta)\zeta
\end{aligned}
$$

Transversality therefore demands

$$
\begin{aligned}
K_0 &= \varphi_1^{\frac{1}{1-\alpha}} \varphi_2^{\frac{\alpha(1-\eta)}{(1-\alpha)\eta}-1} \left[\frac{\mu_0}{(1-\alpha)A_0}\right]^{-\frac{1}{\eta}} \frac{1}{\psi}\frac{1}{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta) \\
&+\underline{C}\varphi_1^{\frac{1}{1-\alpha}} \varphi_2^{\frac{1}{\alpha-1}} \frac{1}{\psi}\frac{1}{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta) \\
&= \frac{C_0-\underline{C}}{\psi}(1-\zeta)^{\tilde{a}_1-1}\frac{1}{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta) \\
&+\frac{\underline{C}}{\psi}(1-\zeta)^{\tilde{a}_1}\frac{1}{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta)
\end{aligned}
\tag{50}
$$

If either $\tilde{b}_1 - 1 > 0$ or $\tilde{b}_2 - 1 > 0$ would not hold, the transversality condition for K_t would be violated because then $\lim_{x_t \to 0} \frac{\partial x_t \lambda_t K_t}{\partial x_t} \to \infty$.

Inserting the transversality condition (50) into (47) and using the definitions for x_t, $\tilde{b}_1$ and $\tilde{b}_2$ gives

$$
\begin{aligned}
K_t &= \frac{C_0-\underline{C}}{\psi} e^{-\delta t} x_t^{-\tilde{a}_2}(1-\zeta x_t)^{\tilde{a}_2}(1-\zeta)^{\tilde{a}_1-\tilde{a}_2}\frac{1}{\tilde{b}_1} x_t^{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta x_t) \\
&+\frac{\underline{C}}{\psi} e^{-\delta t} x_t^{-\tilde{a}_2}(1-\zeta x_t)^{\tilde{a}_2}\frac{1}{\tilde{b}_2} x_t^{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta x_t) \\
&= \frac{C_0-\underline{C}}{\psi} x_t^{(\tilde{a}_1-\tilde{a}_2)\frac{\gamma-\rho}{\gamma+\delta}}(1-\zeta x_t)^{\tilde{a}_2}(1-\zeta)^{\tilde{a}_1-\tilde{a}_2}\frac{1}{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta x_t) \\
&+\frac{\underline{C}}{\psi}(1-\zeta x_t)^{\tilde{a}_2}\frac{1}{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta x_t)
\end{aligned}
$$

As $t \to \infty$, $x_t \to 0$ and we observe for $\tilde{b}_1 - 1 > 0$ and $\tilde{b}_2 - 1 > 0$

$$
\begin{aligned}
\lim_{x_t \to 0} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta x_t) = \tilde{b}_1 \int_0^1 x_z^{\tilde{b}_1-1} dx_z = 1, \\
\lim_{x_t \to 0} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta x_t) = \tilde{b}_2 \int_0^1 x_z^{\tilde{b}_2-1} dx_z = 1.
\end{aligned}
$$

Therefore,

$$\begin{aligned}\lim_{t\to\infty} K_t &= \frac{C_0-\underline{C}}{\psi}(1-\zeta)^{\tilde{a}_1-\tilde{a}_2}\frac{1}{\tilde{b}_1}\lim_{x_t\to 0} x_t^{(\tilde{a}_1-\tilde{a}_2)\frac{\gamma-\rho}{\gamma+\delta}} \\ &\quad +\frac{\underline{C}}{\psi}\frac{1}{\tilde{b}_2},\end{aligned}$$

Depending on the parameter values, we find

$$\lim_{t\to\infty} K_t \begin{cases} = \frac{\underline{C}}{\psi}\frac{1}{\tilde{b}_2} & \text{for } \rho>\gamma, \\ = \frac{C_0-\underline{C}}{\psi}(1-\zeta)^{\tilde{a}_1-\tilde{a}_2}\frac{1}{\tilde{b}_1}+\frac{\underline{C}}{\psi}\frac{1}{\tilde{b}_2} & \text{for } \rho=\gamma, \\ \to\infty & \text{for } \rho<\gamma. \end{cases}$$

F: Transversality condition for S_t:

$$\lim_{t\to\infty}\mu_t S_t = \lim_{t\to\infty}\mu_0 S_t = \mu_0 \lim_{t\to\infty} S_t = 0$$

First, we compute resource extraction over the entire planing horizon. (48) reveals that $\int_0^\infty R_z dz$ only exists if $\tilde{b}_1-1\geq 0$ and $\tilde{b}_2-1\geq 0$ because $\lim_{t\to\infty} x_t = 0$. Rearranging (48) gives

$$\begin{aligned}\int_0^t R_z dz =& \Bigg[-\frac{K_0}{A_0}\varphi_2^{-\tilde{a}_2}(1-\zeta)^{-\tilde{a}_2}\frac{1}{\psi}+\frac{C_0-\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}(1-\zeta)^{\tilde{a}_1-\tilde{a}_2}\frac{1}{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta) \\ &+\frac{\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}\frac{1}{A_0}\frac{1}{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)\Bigg]\left[1-x_t^{-1}\right] \\ &+\frac{C_0-\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}(1-\zeta)^{\tilde{a}_1-\tilde{a}_2}\frac{\zeta^{1-\tilde{b}_1}}{\tilde{b}_1(\tilde{b}_1-1)}\frac{1}{A_0}\Big[\zeta^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)-(\zeta x_t)^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta \\ &+\frac{\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}\frac{\zeta^{1-\tilde{b}_2}}{\tilde{b}_2(\tilde{b}_2-1)}\frac{1}{A_0}\left[\zeta^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)-(\zeta x_t)^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\right],\end{aligned}$$

where we note that the first term in brackets is zero due to the transversality condition (50) for K_t and, hence, we arrive for $t\to\infty$ at

$$\begin{aligned}\int_0^\infty R_z dz =& \frac{C_0-\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}(1-\zeta)^{\tilde{a}_1-\tilde{a}_2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}\frac{1}{A_0}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta) \\ &+\frac{\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}\frac{1}{A_0}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta).\end{aligned} \tag{52}$$

Transversality demands

$$S_0-\int_0^\infty R_z dz = 0, \tag{53}$$

which implies

$$S_0 = \frac{C_0-\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}(1-\zeta)^{\tilde{a}_1-\tilde{a}_2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}\frac{1}{A_0}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta) + \frac{\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}\frac{1}{A_0}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta). \tag{54}$$

Inserting (55) into (51) and using (50) gives

$$S_t = \frac{C_0-\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}(1-\zeta)^{\tilde{a}_1-\tilde{a}_2}\frac{x_t^{\tilde{b}_1-1}}{\tilde{b}_1(\tilde{b}_1-1)}\frac{1}{A_0}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta) + \frac{\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}\frac{x_t^{\tilde{b}_2-1}}{\tilde{b}_2(\tilde{b}_2-1)}\frac{1}{A_0}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta).$$

As long as $\tilde{b}_1-1>0$ and $\tilde{b}_2-1>0$, obviously $\lim_{t\to\infty} S_t = lim_{x_t\to 0}S_t = 0$.

G: Co-state variables: We prove that the transversality conditions (50) and (52) uniquely pin down the initial value of ζ if a solution to the problem exists.

Define

$$\begin{aligned}
K_0^+ &= \frac{C_0-\underline{C}}{\psi}(1-\zeta)^{\tilde{a}_1}\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)}{\tilde{b}_1},\\
\underline{K}_0 &= \frac{\underline{C}}{\psi}(1-\zeta)^{\tilde{a}_2}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)}{\tilde{b}_2},\\
S_0^+ &= \varphi_2^{-\tilde{a}_2}\frac{C_0-\underline{C}}{\psi^2}(1-\zeta)^{\tilde{a}_1-\tilde{a}_2}\frac{1}{A_0}\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)}{\tilde{b}_1(\tilde{b}_1-1)},\\
\underline{S}_0 &= \varphi_2^{-\tilde{a}_2}\frac{\underline{C}}{\psi^2}\frac{1}{A_0}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)}{\tilde{b}_2(\tilde{b}_2-1)}
\end{aligned}$$

Equilibrium demands

$$\frac{K_0-\underline{K}_0}{S_0-\underline{S}_0} = \frac{K_0^+}{S_0^+}, \tag{55}$$

with

$$\frac{K_0^+}{S_0^+} = \psi(\tilde{b}_1-1)A_0\varphi_2^{\tilde{a}_2}(1-\zeta)^{\tilde{a}_2}\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)}{{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)}, \tag{56}$$

$$\frac{K_0-\underline{K}_0}{S_0-\underline{S}_0} = \frac{K_0-\frac{\underline{C}}{\psi}(1-\zeta)^{\tilde{a}_2}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)}{\tilde{b}_2}}{S_0-\frac{\underline{C}}{\psi^2}\varphi_2^{-\tilde{a}_2}\frac{1}{A_0}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)}{\tilde{b}_2(\tilde{b}_2-1)}}.). \tag{57}$$

We first notice that (56) and (57) demand $\zeta<1$.

We show first that $\frac{K_0^+}{S_0^+}$ given by (56) is decreasing in ζ. Second, we show that $\frac{K_0-\underline{K}_0}{S_0-\underline{S}_0}$ given by (57) is increasing in ζ. This implies that there can be at most one solution to (55).

Investigating $\frac{K_0^+}{S_0^+}$, we have to distinguish three cases, i.e. $\tilde{a}_1 < 0, \tilde{a}_1 = 0, \tilde{a}_1 > 0$.

Case 1: $\tilde{a}_1 < 0$: Lemma 1 in Boucekkine and Ruiz-Tamarit (2008) shows that $\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)}{{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)}$ is decreasing in ζ in case $\tilde{a}_1 < 0$. It is obvious that $(1-\zeta)^{\tilde{a}_2}$ is decreasing in ζ as well because $\tilde{a}_2 = \frac{1}{1-\alpha} > 0$. Therefore, $\frac{K_0^+}{S_0^+}$ is in this case decreasing in ζ.

Case 2: $\tilde{a}_1 = 0$: This case prevails if it happens to be that $\eta = \alpha$. Lemma 1 in Boucekkine and Ruiz-Tamarit (2008) shows that in this case $\frac{\partial \frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)}{{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)}}{\partial\zeta} = 0$ applies. As $(1-\zeta)^{\tilde{a}_2}$ is decreasing in ζ, $\frac{K_0^+}{S_0^+}$ is in this case again decreasing in ζ.

Case 3: $\tilde{a}_1 > 0$: The denominator in $\frac{K_0^+}{S_0^+}$ is increasing in ζ as $\frac{\partial\, {}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)}{\partial\zeta} = \frac{\tilde{a}_1(\tilde{b}_1-1)}{\tilde{b}_1+1}{}_2F_1((\tilde{a}_1+1,\tilde{b}_1;\tilde{b}_1+2;\zeta) > 0$ (Abramowitz and Stegun 1972, 15.2.1) because $\tilde{b}_1 - 1 > 0$ is required by the transversality conditions (50) and (55). There are opposing forces at work in the nominator as ${}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)$ increases and $(1-\zeta)^{\tilde{a}_2}$ decreases in ζ. To find out which is stronger, we define $h(\zeta)$ as

$$
\begin{aligned}
h(\zeta) \quad &= \quad (1-\zeta)^{\tilde{a}_2}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta) = (1-\zeta)^{\tilde{a}_2-\tilde{a}_1}(1-\zeta)^{\tilde{a}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta) \\
&\quad\ \text{with} \\
&\quad\ \tilde{a}_2 - \tilde{a}_1 = \frac{1}{1-\alpha} - \frac{\eta-\alpha}{\eta(1-\alpha)} = \frac{\alpha}{\eta(1-\alpha} > 0, \\
&\quad\ {}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta) = \tilde{b}_1\int_0^1 x^{\tilde{b}_1-1}(1-zx)^{-\tilde{a}_1}dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\partial h(\zeta)}{\partial\zeta} &= -(\tilde{a}_2-\tilde{a}_1)\frac{h(\zeta)}{1-\zeta} - \tilde{a}_1\frac{h(\zeta)}{1-\zeta} + (1-\zeta)^{\tilde{a}_2-\tilde{a}_1}(1-\zeta)^{\tilde{a}_1}\frac{\partial\, {}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)}{\partial z} \\
&= -(\tilde{a}_2-\tilde{a}_1)\frac{h(\zeta)}{1-\zeta} - \tilde{a}_1\frac{h(\zeta)}{1-\zeta} + (1-\zeta)^{\tilde{a}_2-\tilde{a}_1}(1-\zeta)^{\tilde{a}_1}\tilde{a}_1\tilde{b}_1\int_0^1 x^{\tilde{b}_1}(1-\zeta x)^{-\tilde{a}_1-1}dx \\
&= -(\tilde{a}_2-\tilde{a}_1)\frac{h(\zeta)}{1-\zeta} + (1-\zeta)^{\tilde{a}_2-\tilde{a}_1}(1-\zeta)^{\tilde{a}_1}\tilde{a}_1\tilde{b}_1\int_0^1\left(x^{\tilde{b}_1}(1-\zeta x)^{-\tilde{a}_1-1} - x^{\tilde{b}_1-1}\frac{(1-\zeta x)^{-\tilde{a}_1}}{1-\zeta}\right)dx \\
&= -(\tilde{a}_2-\tilde{a}_1)\frac{h(\zeta)}{1-\zeta} + (1-\zeta)^{\tilde{a}_2-\tilde{a}_1}(1-\zeta)^{\tilde{a}_1}\tilde{a}_1\tilde{b}_1\int_0^1\left(x^{\tilde{b}_1}(1-\zeta x)^{-\tilde{a}_1-1} - x^{\tilde{b}_1-1}\frac{(1-\zeta x)^{-\tilde{a}_1}}{1-\zeta}\right)dx \\
&= -(\tilde{a}_2-\tilde{a}_1)\frac{h(\zeta)}{1-\zeta} + (1-\zeta)^{\tilde{a}_2-\tilde{a}_1}(1-\zeta)^{\tilde{a}_1}\tilde{a}_1\tilde{b}_1\int_0^1\left(x^{\tilde{b}_1-1}(1-\zeta x)^{-\tilde{a}_1-1}\left(x - \frac{1-\zeta x}{1-\zeta}\right)\right)dx \\
&= -(\tilde{a}_2-\tilde{a}_1)\frac{h(\zeta)}{1-\zeta} + (1-\zeta)^{\tilde{a}_2-\tilde{a}_1}(1-\zeta)^{\tilde{a}_1}\tilde{a}_1\tilde{b}_1\int_0^1\left(x^{\tilde{b}_1-1}(1-\zeta x)^{-\tilde{a}_1-1}\frac{x-1}{1-\zeta}\right)dx \\
&= -(\tilde{a}_2-\tilde{a}_1)\frac{h(\zeta)}{1-\zeta} - \tilde{a}_1(1-\zeta)^{\tilde{a}_2-1}\tilde{b}_1\int_0^1 x^{\tilde{b}_1-1}(1-x)(1-\zeta x)^{-\tilde{a}_1-1}dx \\
&= -(\tilde{a}_2-\tilde{a}_1)\frac{h(\zeta)}{1-\zeta} - \tilde{a}_1(1-\zeta)^{\tilde{a}_2-1}\frac{{}_2F_1(\tilde{a}_1+1,\tilde{b}_1;\tilde{b}_1+2;\zeta)}{\tilde{b}_1+1}.
\end{aligned}
$$

As $\tilde{a}_2-\tilde{a}_1>0$ and $\tilde{a}_1>0$ in this case, we find $\frac{\partial h(\zeta)}{\partial \zeta}<0$. Summing up case 3, the denominator in $\frac{K_0^+}{S_0^+}$ is increasing while the nominator is decreasing in ζ. Hence, $\frac{K_0^+}{S_0^+}$ is again decreasing in ζ.

We turn to $\frac{K_0-\underline{K}_0}{S_0-\underline{S}_0}$ given by (57). Its denominator is obviously decreasing in ζ as $\tilde{a}_2=\frac{1}{1-\alpha}>0$ and $\frac{\partial\, {}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)}{\partial\zeta}=\frac{\tilde{a}_2(\tilde{b}_2-1)}{\tilde{b}_2+1}\,{}_2F_1(\tilde{a}_2+1,\tilde{b}_2;\tilde{b}_2+2;\zeta)$ with $\tilde{b}_2-1>0$ due to the transversality condition (55).

The nominator in $\frac{K_0-\underline{K}_0}{S_0-\underline{S}_0}$ is increasing in ζ. To see this, define

$$k(\zeta) \;=\; (1-\zeta)^{\tilde{a}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)=(1-\zeta)^{\tilde{a}_2}\tilde{b}_2\int_0^1 x^{\tilde{b}_2-1}(1-\zeta x)^{-\tilde{a}_2}dx.$$

Therefore,

$$\begin{aligned}
\frac{\partial k(\zeta)}{\partial\zeta} &= \tilde{a}_2(1-\zeta)^{\tilde{a}_2}\tilde{b}_2\left[\int_0^1 x^{\tilde{b}2}(1-\zeta x)^{-\tilde{a}_2-1}dx-\int_0^1 x^{\tilde{b}2-1}\frac{(1-\zeta x)^{-\tilde{a}_2}}{1-\zeta}dx\right]\\
&= \tilde{a}_2(1-\zeta)^{\tilde{a}_2}\tilde{b}_2\int_0^1 x^{\tilde{b}_2-1}(1-\zeta x)^{-\tilde{a}_2-1}\left[x-\frac{1-\zeta x}{1-\zeta}\right]dx\\
&= -\tilde{a}_2(1-\zeta)^{\tilde{a}_2-1}\tilde{b}_2\int_0^1 x^{\tilde{b}_2-1}(1-x)(1-\zeta x)^{-\tilde{a}_2-1}dx\\
&= -\tilde{a}_2(1-\zeta)^{\tilde{a}_2-1}\frac{{}_2F_1(\tilde{a}_2+1,\tilde{b}_2;\tilde{b}_2+2;\zeta)}{\tilde{b}_2+1}
\end{aligned}$$

which is negative for $\zeta<1$.

Summing up, we have shown that the left hand side of (55) is increasing while the right hand side is decreasing in ζ. If an equilibrium fulfilling (55) exists, it is unique.

Properties of $\frac{K_0^+}{S_0^+}$: To work out conditions for existence, we focus first on $\frac{K_0^+}{S_0^+}$ given by (56). Any solution ζ needs to fulfill $\zeta<1$; we know that $\frac{K_0^+}{S_0^+}$ is decreasing in ζ. We show first that $\frac{K_0^+}{S_0^+}$ is unbounded from above for $\zeta\to-\infty$. Let ε_1 be an arbitrarily large but finite real number. The critical term in $\frac{K_0^+}{S_0^+}$ is $(1-\zeta)^{\tilde{a}_2}\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)}{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)}$. Now suppose that

$$\lim_{\zeta\to-\infty}(1-\zeta)^{\tilde{a}_2}\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)}{{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)}<\varepsilon_1 \tag{58}$$

would be true. As $\frac{K_0^+}{S_0^+}$ decreases with ζ. This would imply that for any finite $\zeta<1$ and for $\zeta\to-\infty$ it would be true that

$$\begin{aligned}
{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta) &< \varepsilon_1(1-\zeta)^{-\tilde{a}_2}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta),\\
\tilde{b}_1\int_0^1 x^{\tilde{b}_1-1}(1-\zeta x)^{-\tilde{a}_1}dx &- \varepsilon_1(1-\zeta)^{-\tilde{a}_2}\tilde{b}_1(\tilde{b}_1-1)\int_0^1 x^{\tilde{b}_1-2}(1-x)(1-\zeta x)^{-\tilde{a}_1}dx<0,\\
\int_0^1 x^{\tilde{b}_1-2}(1-\zeta x)^{-\tilde{a}_1} & \left[x-\varepsilon_1(\tilde{b}_1-1)(1-\zeta)^{-\tilde{a}_2}(1-x)\right]dx<0,\\
\int_0^1 x^{\tilde{b}_1-2}(1-\zeta x)^{-\tilde{a}_1} & \kappa(x;\varepsilon_1)dx<0, \qquad (59)\\
\text{with} & \\
\kappa(x;\varepsilon_1) &= \left[x-\varepsilon_1(\tilde{b}_1-1)(1-\zeta)^{-\tilde{a}_2}(1-x)\right],
\end{aligned}$$

where $\kappa(x;\varepsilon_1)$ is an affine function of x. $\kappa(x;\varepsilon_1)$ is zero for $x=\bar{x}_0(\varepsilon_1)$ with

$$\bar{x}_0(\varepsilon_1)=\frac{\varepsilon_1(\tilde{b}_1-1)(1-\zeta)^{-\tilde{a}_2}}{1+\varepsilon_1(\tilde{b}_1-1)(1-\zeta)^{-\tilde{a}_2}}. \qquad (60)$$

Therefore, $\kappa(x;\varepsilon_1)<0$ for $x<\bar{x}_0(\varepsilon_1)$ and $\kappa(x;\varepsilon_1)>0$ for $x>\bar{x}_0(\varepsilon_1)$. For any finite ε_1, $\bar{x}_0(\varepsilon_1)\to 0$ for $\zeta\to-\infty$ as $\tilde{a}_2=\frac{1}{1-\alpha}>0$. As we integrate from 0 to 1, $\kappa(x;\varepsilon_1)$ becomes positive for $0\le x\le 1$ as $\zeta\to-\infty$ and inequality (59) cannot be fulfilled. Hence, $\frac{K_0^+}{S_0^+}$ cannot be bounded from above as $\zeta\to-\infty$ and $\lim_{\zeta\to-\infty}\frac{K_0^+}{S_0^+}=\infty$.

Next, turn to the case $\zeta\to 1$. Suppose that $\frac{K_0^+}{S_0^+}$ would be bounded from below by some $\varepsilon_2>0$. By the same logic as above, this would imply for any $\zeta<1$ and $\zeta\to 1$ that

$$\begin{aligned}
{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta) &> \varepsilon_2(1-\zeta)^{-\tilde{a}_2}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta),\\
\int_0^1 x^{\tilde{b}_1-2}(1-\zeta x)^{-\tilde{a}_1} & \kappa(x;\varepsilon_2)dx>0. \qquad (61)
\end{aligned}$$

For any finite $\varepsilon_2>0$, $\bar{x}_0(\varepsilon_2)\to 1$ for $\zeta\to 1$ as $\tilde{a}_2=\frac{1}{1-\alpha}>0$. As we integrate from 0 to 1, $\kappa(x;\varepsilon_2)$ becomes negative for $0\le x\le 1$ as $\zeta\to 1$ and inequality (61) cannot be fulfilled. Hence, $\frac{K_0^+}{S_0^+}$ cannot be bounded from below and $\lim_{\zeta\to 1}\frac{K_0^+}{S_0^+}=0$.

Properties of $\frac{K_0-\underline{K}_0}{S_0-\underline{S}_0}$ as $\zeta\to 1$: We turn to $\frac{K_0-\underline{K}_0}{S_0-\underline{S}_0}$ which we know is increasing in ζ for $\zeta<1$. If a maximum exists, it must be reached as $\zeta\to 1$. The critical term in the nominator is $(1-\zeta)^{\tilde{a}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)$ which can be written as $(1-\zeta)\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)}{(1-\zeta)^{1-\tilde{a}_2}}$. We are interested in

$$\lim_{\zeta\to 1}(1-\zeta)\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)}{(1-\zeta)^{1-\tilde{a}_2}} \qquad (62)$$

as $\lim_{\zeta\to 1}(1-\zeta)$ is finite and equal to zero, we can rewrite this expression as

$$\lim_{\zeta\to 1}(1-\zeta)\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)}{(1-\zeta)^{1-\tilde{a}_2}} = \left[\lim_{\zeta\to 1}(1-\zeta)\right]\left[\lim_{\zeta\to 1}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)}{(1-\zeta)^{1-\tilde{a}_2}}\right] \tag{63}$$

if the second limit on the right hand side in the above equation is finite. 15.4.23 in DLMF (URL) states that

$$\lim_{\zeta\to 1}\frac{{}_2F_1(a,b;c;z)}{(1-z)^{c-a-b}} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \tag{64}$$

if $\Re(c-a-b)<0$. Applied to our case, $c-a-b = 1+\tilde{b}_2-\tilde{a}_2-\tilde{b}_2 = 1-\tilde{a}_2 = -\frac{\alpha}{1-\alpha}<0$. Furthermore, $\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} = \frac{\Gamma(\tilde{b}_2+1)\Gamma(\tilde{a}_2-1)}{\Gamma(\tilde{a}_2)\Gamma(\tilde{b}_2)} = \frac{\tilde{b}_2}{\tilde{a}_2-1}$ which is finite. Hence, $\lim_{\zeta\to 1}(1-\zeta)\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)}{(1-\zeta)^{1-\tilde{a}_2}}=0$ and $\lim_{\zeta\to 1}K_0-\underline{K}_0 = K_0$.

The critical term in the denominator of $\frac{K_0-\underline{K}_0}{S_0-\underline{S}_0}$ is ${}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)$. As $\frac{\partial\,{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)}{\partial\zeta} = \frac{\tilde{a}_2(\tilde{b}_2-1)}{\tilde{b}_2+1}{}_2F_1(\tilde{a}_2+1,\tilde{b}_2;\tilde{b}_2+2;\zeta)>0$ for $\zeta<1$, $S_0-\underline{S}_0$ declines with ζ in this range. 15.3.6 in Abramowitz and Stegun (1972) implies that $\lim_{\zeta\to 1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta) = \frac{\Gamma(\tilde{b}_2+1)\Gamma(2-\tilde{a}_2)}{\Gamma(\tilde{b}_2+1-\tilde{a}_2)\Gamma(2)}$ if $2-\tilde{a}_2 = \frac{1-2\alpha}{1-\alpha}>0$ which is the case for $\alpha<\frac{1}{2}$. In case $\alpha>\frac{1}{2}$ we find ${}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)\to\infty$ as $\zeta\to 1$. In both cases, it is possible that $S_0-\underline{S}_0$ turns negative as ζ grows for $\zeta<1$. Define $\bar{\zeta}$ as

$$\bar{\zeta} =_{\zeta\le 1} |S_0 - \frac{\underline{C}}{A_0}\varphi_2^{-\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)|, \tag{65}$$

As $S_0-\underline{S}_0$ is decreasing in ζ for $\zeta<1$, the admissible range for a solution to the present problem has the upper bound $\bar{\zeta}$. Therefore, if $\bar{\zeta}<1$ ($\bar{\zeta}=1$) we find $S_0-\underline{S}_0|_{\zeta=\bar{\zeta}}=0$ ($S_0-\underline{S}_0|_{\zeta=\bar{\zeta}}\ge 0$).

Lastly, we turn to $\frac{K_0-\underline{K}_0}{S_0-\underline{S}_0}$ as $\zeta\to-\infty$. Again, we start with the nominator $K_0-\underline{K}_0 = K_0-\underline{C}\frac{1}{\psi}(1-\zeta)^{\tilde{a}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)$. We know already that $(1-\zeta)^{\tilde{a}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)$ is decreasing in ζ for $\zeta<1$. Obviously, $K_0-\underline{K}_0$ then declines as $\zeta\to-\infty$. 15.3.4 in Abramowitz and Stegun (1972) states that

$${}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1(a,c-b;c;\frac{z}{z-1}) \tag{66}$$

which implies for the present case

$$(1-\zeta)^{\tilde{a}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta) = {}_2F_1(\tilde{a}_2,1;\tilde{b}_2+1;\frac{\zeta}{\zeta-1}). \tag{67}$$

As $\tilde{a}_2,\tilde{b}_2+1>0$ and $\lim_{\zeta\to-\infty}\frac{\zeta}{\zeta-1}=1$, $\lim_{\zeta\to-\infty}(1-\zeta)^{\tilde{a}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)=\infty$. This implies that $K_0-\underline{K}_0$ becomes necessarily negative if ζ becomes too small. The range for admissible values for ζ is therefore bounded from below at $\underline{\zeta}$ which satisfies the condition

$$K_0 = \underline{C}\frac{1}{\psi}\frac{1}{\tilde{b}_2}(1-\underline{\zeta})^{\tilde{a}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\underline{\zeta}). \tag{68}$$

We observe $\lim_{\zeta\to\underline{\zeta}}\frac{K_0-\underline{K}_0}{S_0-\underline{S}_0}=0$.

Taken together, if $\underline{\zeta}<\bar{\zeta}$ and $\bar{\zeta}<1$, $\lim_{\zeta\to\bar{\zeta}}\frac{K_0-\underline{K}_0}{S_0-\underline{S}_0}\to\infty$. If if $\underline{\zeta}<\bar{\zeta}$, $\bar{\zeta}=1$,$\lim_{\zeta\to\bar{\zeta}}\frac{K_0-\underline{K}_0}{S_0-\underline{S}_0}$ either diverges to infinity or a strictly positive constant. The latter occurs if $S_0-\underline{S}_0\neq 0$ for $\zeta\le 1$ or if $\alpha<\frac{1}{2}$ and $S_0-\underline{S}_0=0$ for $\zeta=1$. In all possible cases we therefore observe $\lim_{\zeta\to\bar{\zeta}}\frac{K_0-\underline{K}_0}{S_0-\underline{S}_0}>\lim_{\zeta\to\bar{\zeta}}\frac{K_0^+}{S_0^+}$.

Furthermore, if $\underline{\zeta}<\bar{\zeta}$ we know that $\lim_{\zeta\to\underline{\zeta}}\frac{K_0-\underline{K}_0}{S_0-\underline{S}_0}=0$ and $\lim_{\zeta\to\underline{\zeta}}\frac{K_0^+}{S_0^+}>0$ as $\frac{K_0^+}{S_0^+}$ is decreasing in ζ for

$\zeta < 1$ and approaches 0 as $\zeta \to 1$.

If it happens that $\underline{\zeta} = \bar{\zeta}$, this value is the unique solution to the initial value problem. If we find $\underline{\zeta} > \bar{\zeta}$, there is no solution to the initial value problem because initial endowments K_0, S_0 are too low to allow for subsistence consumption $\underline{C}$.

This proves that a unique solution always exits if and only if $\underline{\zeta} \leq \bar{\zeta}$.

H: Sustainability present values: $PV[X_s]_t$ denotes the present value of X_s at time t for $s \in [t, \infty)$. Discounting uses the interest rate r_s given in (35). We start by evaluating the present value of a consumption stream $\bar{C}_t - \underline{C}$ that is constant from t onwards:

$$PV(\bar{C}_t - \underline{C})_t = \int_t^\infty (\bar{C}_t - \underline{C}) e^{-\int_t^s r_\tau d\tau} ds$$

With r_τ given by (35), $x_\tau = e^{-\psi\tau}$, and $\psi = \frac{1-\alpha}{\alpha}(\gamma + \delta)$

$$\begin{aligned}
-\int_t^s r_\tau d\tau &= -\int_t^s (\gamma+\delta)(1-\zeta_0 x_\tau)^{-1} - \delta d\tau \\
&= (\gamma+\delta)\int_t^s \frac{1}{\psi}(1-\zeta_0 x_\tau)^{-1} x_\tau^{-1} dx_\tau + \delta(s-t) \\
&= \frac{\alpha}{1-\alpha}\int_{x_t}^{x_s} (1-\zeta_0 x_\tau)^{-1} x_\tau^{-1} dx_\tau + \delta(s-t) \\
&= \frac{\alpha}{1-\alpha}\left[\ln \frac{x_\tau}{1-x_\tau}\right]_{x_t}^{x_s} + \delta(s-t).
\end{aligned}$$

Therefore,

$$e^{-\int_t^s r_\tau d\tau} = \left(\frac{x_s}{x_t}\frac{1-\zeta_0 x_t}{1-\zeta_0 x_s}\right)^{\frac{\alpha}{1-\alpha}} e^{\delta(s-t)} \tag{69}$$

and

$$PV(\bar{C}_t - \underline{C})_t = (\bar{C}_t - \underline{C}) e^{-\delta t} x_t^{-\frac{\alpha}{1-\alpha}} (1-\zeta_0 x_t)^{\frac{\alpha}{1-\alpha}} \int_t^\infty x_s^{\frac{\alpha}{1-\alpha}} (1-\zeta_0 x_s)^{-\frac{\alpha}{1-\alpha}} e^{\delta s} ds.$$

Using $x_\tau = e^{-\psi\tau}$, defining $x = \frac{x_s}{x_t}$ and noting that $dx = x_t^{-1} dx_s = -\psi x_t^{-1} x_s ds$ gives

$$\begin{aligned}
PV(\bar{C}_t - \underline{C})_t &= (\bar{C}_t - \underline{C})(1-\zeta_0 x_t)^{\frac{\alpha}{1-\alpha}} \frac{1}{\psi}\int_0^1 x^{\frac{\alpha}{1-\alpha}\frac{\gamma}{\delta+\gamma}-1}(1-\zeta_0 x_t x)^{-\frac{\alpha}{1-\alpha}} dx \\
&= (\bar{C}_t - \underline{C})(1-\zeta_0 x_t)^{\frac{\alpha}{1-\alpha}} \frac{1}{\psi(\tilde{b}_2 - 1)} {}_2F_1(\tilde{a}_2 - 1, \tilde{b}_2 - 1; \tilde{b}_2; \zeta_0 x_t),
\end{aligned}$$

where $\tilde{a}_2$ and $\tilde{b}_2$ are defined as in (19).

The present value of welfare-maximizing consumption in excess of $\underline{C}$ is

$$PV(C_s-\underline{C})_t = \int_t^\infty (C_s-\underline{C})e^{-\int_t^s r_\tau d\tau}ds,$$

where $C_s-\underline{C}$ is given by (32). Using (69 and applying again the definitions from above gives

$$\begin{aligned}
PV(C_s-\underline{C})_t &= \varphi_2^{\frac{\alpha}{(1-\alpha)\eta}}\left[\frac{\mu_0}{(1-\alpha)A_0}\right]^{-\frac{1}{\eta}}\int_t^\infty x_s^{\frac{\alpha(\rho-\gamma)}{(1-\alpha)(\gamma+\delta)\eta}}(1-\zeta_0 x_s)^{\frac{\alpha}{1-\alpha}\frac{1}{\eta}}e^{\delta s}\left(\frac{x_s}{x_t}\frac{1-\zeta_0 x_t}{1-\zeta_0 x_s}\right)^{\frac{\alpha}{1-\alpha}}e^{\delta(s-t)}ds \\
&= \frac{C_0-\underline{C}}{\psi}(1-\zeta_0)^{\tilde{a}_1-\tilde{a}_2}x_t^{(\tilde{a}_1-\tilde{a}_2)\frac{\gamma-\rho}{\gamma+\delta}}(1-\zeta_0 x_t)^{\frac{\alpha}{1-\alpha}}\int_0^1 x^{(\tilde{b}_1-1)-1}(1-\zeta_0 x_t x)^{-(\tilde{a}_1-1)}dx \\
&= \frac{C_0-\underline{C}}{\psi}(1-\zeta_0)^{\tilde{a}_1-\tilde{a}_2}x_t^{(\tilde{a}_1-\tilde{a}_2)\frac{\gamma-\rho}{\gamma+\delta}}(1-\zeta_0 x_t)^{\frac{\alpha}{1-\alpha}}\frac{1}{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1-1,\tilde{b}_1-1;\tilde{b}_1;\zeta_0 x_t)
\end{aligned}$$

Equating $PV(\bar{C}_t-\underline{C})_t$ and $PV(C_s-\underline{C})_t$ gives $\bar{C}_t-\underline{C}$ as

$$\begin{aligned}
\bar{C}_t-\underline{C} &= \varphi_2^{\frac{\alpha}{(1-\alpha)\eta}}\left[\frac{\mu_0}{(1-\alpha)A_0}\right]^{-\frac{1}{\eta}}\frac{\tilde{b}_2-1}{\tilde{b}_1-1}x_t^{\frac{\alpha(\rho-\gamma)}{(1-\alpha)(\gamma+\delta)\eta}}\frac{{}_2F_1(\tilde{a}_1-1,\tilde{b}_1-1;\tilde{b}_1;\zeta_0 x_t)}{{}_2F_1(\tilde{a}_2-1,\tilde{b}_2-1;\tilde{b}_2;\zeta_0 x_t)} \\
&= (C_0-\underline{C})(1-\zeta_0)^{\tilde{a}_1-\tilde{a}_2}\frac{\tilde{b}_2-1}{\tilde{b}_1-1}\frac{{}_2F_1(\tilde{a}_1-1,\tilde{b}_1-1;\tilde{b}_1;\zeta_0 x_t)}{{}_2F_1(\tilde{a}_2-1,\tilde{b}_2-1;\tilde{b}_2;\zeta_0 x_t)}x_t^{(\tilde{a}_1-\tilde{a}_2)\frac{\gamma-\rho}{\gamma+\delta}},
\end{aligned} \tag{70}$$

where we used the definition of $\zeta_0 = 1-\frac{\varphi_1}{\varphi_2}$ and the first order condition for C_0 from (6) at $t=0$. We note that $PV(\bar{C}_t-\underline{C})_t$ and $PV(C_s-\underline{C})_t$ are in general depending on t. For $t\to\infty$ and, hence, $x_t\to 0$ we arrive at the steady state. In steady state, the interest rate is $r=\gamma$ and the growth rate of consumption is $g=\frac{1}{\eta}(\gamma-\rho)$. The present values become

$$\begin{aligned}
\lim_{t\to\infty} PV(\bar{C}_t-\underline{C})_t &= \lim_{t\to\infty}(\bar{C}_t-\underline{C})(1-\zeta_0 x_t)^{\frac{\alpha}{1-\alpha}}\frac{1}{\psi(\tilde{b}_2-1)}{}_2F_1(\tilde{a}_2-1,\tilde{b}_2-1;\tilde{b}_2;\zeta_0 x_t) \\
&= \frac{\lim_{t\to\infty}(\bar{C}_t-\underline{C})}{\gamma} = \frac{\lim_{t\to\infty}(\bar{C}_t-\underline{C})}{r}, \\
\lim_{t\to\infty} PV(C_s-\underline{C})_t &= \lim_{t\to\infty}\frac{C_0-\underline{C}}{\psi}(1-\zeta_0)^{\tilde{a}_1-\tilde{a}_2}\varphi_2^{\frac{\alpha}{(1-\alpha)\eta}}\left[\frac{\mu_0}{(1-\alpha)A_0}\right]^{-\frac{1}{\eta}}x_t^{(\tilde{a}_1-\tilde{a}_2)\frac{\gamma-\rho}{\gamma+\delta}}(1-\zeta_0 x_t)^{\frac{\alpha}{1-\alpha}}\times \\
&\quad \frac{1}{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1-1,\tilde{b}_1-1;\tilde{b}_1;\zeta_0 x_t) \\
&= (1-\zeta_0)^{1-\tilde{a}_1}\frac{C_0-\underline{C}}{r-g}\lim_{t\to\infty}x_t^{(\tilde{a}_1-\tilde{a}_2)\frac{\gamma-\rho}{\gamma+\delta}}.
\end{aligned}$$

We note that the steady state present value of optimal consumption depends on the initial conditions in our economy. Equating the present values as $t\to\infty$ gives

$$\lim_{t\to\infty} \bar{C}_t - \underline{C} \quad = \quad \begin{cases} 0 & \text{for } \rho > \gamma, \\ (1-\zeta_0)^{1-\tilde{a}_1}(C_0 - \underline{C})\frac{r}{r-g} & \text{for } \rho = \gamma, \\ (1-\zeta_0)^{1-\tilde{a}_1}(C_0 - \underline{C})\frac{r}{r-g} \lim_{t\to\infty} x_t^{(\tilde{a}_1-\tilde{a}_2)\frac{\gamma-\rho}{\gamma+\delta}} \to \infty & \text{for } \rho < \gamma. \end{cases}$$

Now we compute the maximum affordable consumption $\underline{C}_t^{max}$ from time $t \geq 0$ onward while the economy was following the welfare-maximizing consumption path for $s \in [0,t)$. (33) gives K_t as the available capital endowment in t. To arrive at $\underline{C}_t^{max}$, we use (33) once again at $C_t - \underline{C} = 0$ with $C_t - \underline{C} = (C_0 - \underline{C})(1-\zeta)^{\tilde{a}_1-\tilde{a}_2} x_t^{\frac{\gamma-\rho}{\gamma+\delta}(\tilde{a}_1-\tilde{a}_2)}(1-\zeta x_t)^{\tilde{a}_2-\tilde{a}_1}$ from (32) and $\underline{C} = \underline{C}_t^{max}$. It follows that $\underline{C}_t^{max} = \psi K_t (1 - \zeta_0 x_t)^{-\tilde{a}_2} \frac{\tilde{b}_2}{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta_0 x_t)}$ which can be written by using (33) as

$$\underline{C}_t^{max} - \underline{C} = (1-\zeta_0)^{\tilde{a}_1-\tilde{a}_2}(C_0 - \underline{C})\frac{\tilde{b}_2}{\tilde{b}_1}\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta_0 x_t)}{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta_0 x_t)} x_t^{(\tilde{a}_1-\tilde{a}_2)\frac{\gamma-\rho}{\gamma+\delta}}.$$

$\underline{C}_0^{max}$, the attainable maximum constant consumption from $t = 0$ onwards can be found be using (33) for $\eta \to \infty$ and, hence, $\tilde{a}_1 \to \tilde{a}_2$, $\tilde{b}_1 \to \tilde{b}_2$

$$\underline{C}_0^{max} = \psi K_0 (1-\zeta_0)^{-\tilde{a}_2} \frac{\tilde{b}_2}{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta_0)}.$$

Dividing this by (70) gives (40) in the main text. As $t \to \infty$, $x_t \to 0$ and ${}_2F_1(a,b;b+1;zx_t) \to 1$ (see Appendix E). This gives

$$\lim_{t\to\infty} \frac{\underline{C}_t^{max} - \underline{C}}{\bar{C}_t - \underline{C}} = \frac{\tilde{b}_1 - 1}{\tilde{b}_1}\frac{\tilde{b}_2}{\tilde{b}_2 - 1}.$$

We observe that

$$\begin{aligned} \frac{\tilde{b}_1 - 1}{\tilde{b}_2 - 1} &= 1 + \frac{\alpha(\rho-\gamma)}{\alpha\eta\gamma}, \\ \frac{\tilde{b}_2}{\tilde{b}_1} &= \left(1 + \frac{\alpha(\rho-\gamma)}{\alpha\eta\gamma + (1-\alpha)\eta(\gamma+\rho)}\right)^{-1}. \end{aligned}$$

Given the admissible parameter values, $\frac{\tilde{b}_1-1}{\tilde{b}_1}\frac{\tilde{b}_2}{\tilde{b}_2-1} > (=,<) 1$ for $\gamma < (=,>) \rho$.

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